



TFY 4250 / FY 2045 - QUM 1

Final Exam 2013

- Solutions -

Note: Each of the 6 problems is worth 10 marks.

=> Total: 60 marks

Solutions

1) We compute

$$\frac{d}{dt} \langle F \rangle = \frac{d}{dt} \int \psi^* \hat{F} \psi dx$$

$$= \int \left[\left(\frac{\partial \psi^*}{\partial t} \right) \hat{F} \psi + \psi^* \frac{\partial \hat{F}}{\partial t} \psi + \psi^* \hat{F} \frac{\partial \psi}{\partial t} \right] dx$$

Using SE:
 $i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$

$$= \frac{i}{\hbar} \int \left((\hat{H} \psi)^* \hat{F} \psi + \psi^* \frac{\partial \hat{F}}{\partial t} \psi - \psi^* \hat{F} \hat{H} \psi \right) dx$$

 $\hat{H}$ Hermitian

$$= \frac{i}{\hbar} \int \psi^* \hat{H} \hat{F} \psi dx + \int \psi^* \frac{\partial \hat{F}}{\partial t} \psi dx - \frac{i}{\hbar} \int \psi^* \hat{F} \hat{H} \psi dx$$

$$= \frac{i}{\hbar} \int \psi^* [\hat{H}, \hat{F}] \psi dx + \int \psi^* \frac{\partial \hat{F}}{\partial t} \psi dx$$

$$= \frac{i}{\hbar} \langle [\hat{H}, \hat{F}] \rangle + \langle \frac{\partial \hat{F}}{\partial t} \rangle \quad \square$$

2) a) We compute $\langle H \rangle$ according to

$$\langle H \rangle = \frac{\int_0^{\infty} \psi^*(x) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x) dx}{\int_0^{\infty} \psi^*(x) \psi(x) dx}$$

Let us start with the denominator:

$$\begin{aligned} \int_0^{\infty} \psi^*(x) \psi(x) dx &= |A|^2 \int_0^{\infty} x^2 e^{-2ax} dx \\ &= |A|^2 \left(\underbrace{-\frac{1}{2a} e^{-2ax} x^2}_{=0} \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{2a} e^{-2ax} 2x dx \right) \\ &= |A|^2 \frac{2}{2a} \left(\underbrace{-\frac{1}{2a} e^{-2ax} x}_{=0} \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{2a} e^{-2ax} dx \right) \\ &= \frac{|A|^2}{a} \left(-\frac{1}{4a^2} e^{-2ax} \Big|_0^{\infty} \right) \\ &= \frac{|A|^2}{4a^3} \end{aligned}$$

The numerator turns into

$$\begin{aligned} &|A|^2 \int_0^{\infty} x e^{-ax} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \gamma x \right) x e^{-ax} dx \\ &= |A|^2 \int_0^{\infty} x e^{-ax} \left(\frac{\hbar^2}{2m} (2a - a^2 x) e^{-ax} + \gamma x^2 e^{-ax} \right) dx \\ &= |A|^2 \int_0^{\infty} e^{-2ax} \left(\frac{\hbar^2}{2m} (2ax - a^2 x^2) + \gamma x^3 \right) dx \\ &= |A|^2 \left[\underbrace{\int_0^{\infty} e^{-2ax} \left(\frac{\hbar^2}{2m} 2ax \right) dx}_{(1)} - \underbrace{\int_0^{\infty} e^{-2ax} \left(+\frac{\hbar^2}{2m} a^2 x^2 \right) dx}_{(2)} + \underbrace{\int_0^{\infty} e^{-2ax} \gamma x^3 dx}_{(3)} \right] \end{aligned}$$

$$\begin{aligned} \left(\frac{d}{dx} (x e^{-ax}) \right) &= (1 - ax) e^{-ax} \\ \Rightarrow \frac{d^2}{dx^2} (x e^{-ax}) &= (-2a + a^2 x) e^{-ax} \end{aligned}$$

Integral ①:
$$\int_0^{\infty} \frac{t^2}{m} ax e^{-2ax} dx = \underbrace{-\frac{1}{2a} e^{-2ax} \frac{t^2}{m} ax \Big|_0^{\infty}}_{=0} + \int_0^{\infty} \frac{t^2}{2m} e^{-2ax} dx$$

$$= -\frac{1}{2a} \left(\frac{t^2}{2m} \right) e^{-2ax} \Big|_0^{\infty}$$

$$= \frac{t^2}{4am}$$

Integral ②:
$$\int_0^{\infty} \frac{t^2}{2m} a^2 x^2 e^{-2ax} dx = \underbrace{-\frac{1}{2a} e^{-2ax} \frac{t^2}{2m} a^2 x^2 \Big|_0^{\infty}}_{=0} + \int_0^{\infty} \frac{t^2}{2m} ax e^{-2ax} dx$$

$$= \underbrace{-\frac{1}{2a} e^{-2ax} \frac{t^2}{2m} ax \Big|_0^{\infty}}_{=0} + \int_0^{\infty} \frac{t^2}{4m} e^{-2ax} dx$$

$$= -\frac{t^2}{8am} e^{-2ax} \Big|_0^{\infty} = \frac{t^2}{8am}$$

Integral ③:
$$\int_0^{\infty} \gamma x^3 e^{-2ax} dx = \underbrace{-\frac{1}{2a} e^{-2ax} \gamma x^3 \Big|_0^{\infty}}_{=0} + \int_0^{\infty} \frac{3\gamma}{2a} x^2 e^{-2ax} dx$$

$$= \underbrace{-\frac{3}{4a^2} \gamma x^2 e^{-2ax} \Big|_0^{\infty}}_{=0} + \int_0^{\infty} \frac{3\gamma}{2a^2} x e^{-2ax} dx$$

$$= \underbrace{-\frac{3\gamma}{4a^3} x e^{-2ax} \Big|_0^{\infty}}_{=0} + \int_0^{\infty} \frac{3\gamma}{4a^3} e^{-2ax} dx$$

$$= -\frac{3\gamma}{8a^4} e^{-2ax} \Big|_0^{\infty} = \frac{3\gamma}{8a^4}$$

Hence, we obtain

$$\langle H \rangle = \frac{\frac{\hbar^2}{4am} - \frac{\hbar^2}{8am} + \frac{3\gamma}{8a^4}}{\frac{1}{4a^3}}$$

($|A|^2$ cancels out)

$$\Rightarrow \boxed{\langle H \rangle = \frac{\hbar^2}{2m} a^2 + \frac{3\gamma}{2a}} \quad (*)$$

b) We need $\psi(0) = 0$ and $\psi(x) \xrightarrow{x \rightarrow \infty} 0$.

$\psi(x)$ fulfills these criteria.

c) Now, we need to minimize $\langle H \rangle$ with respect to a .

$$\Rightarrow \frac{d\langle H \rangle}{da} \stackrel{!}{=} 0$$

$$\Rightarrow \frac{\hbar^2}{m} a - \frac{3\gamma}{2a^2} = 0$$

$$\Rightarrow \frac{\hbar^2}{m} a^3 = \frac{3\gamma}{2}$$

$$\Rightarrow a = \left(\frac{3\gamma m}{2\hbar^2} \right)^{1/3}$$

Substitution into (*) gives

$$\begin{aligned} \langle H \rangle_{\varphi}^{\min} &= \frac{\hbar^2}{2m} \left(\frac{3\gamma m}{2\hbar^2} \right)^{2/3} + \frac{3}{2} \gamma \left(\frac{2\hbar^2}{3\gamma m} \right)^{1/3} \\ &= \frac{3}{4} \left(\frac{2\gamma^2 \hbar^2}{3m} \right)^{1/3} + \frac{3}{2} \left(\frac{2\gamma^2 \hbar^2}{3m} \right)^{1/3} \end{aligned}$$

$$\Rightarrow \boxed{\langle H \rangle_{\varphi}^{\min} = \frac{9}{4} \left(\frac{2\gamma^2 \hbar^2}{3m} \right)^{1/3}} \geq E_0$$

3) a) The energy corrections at first order are:

$$E_n^{(1)} = \langle n | H_1 | n \rangle = \langle \psi_n^{(0)} | H_1 | \psi_n^{(0)} \rangle$$

$$= -F \langle \psi_n^{(0)} | x | \psi_n^{(0)} \rangle = 0, \text{ according to}$$

$$\Rightarrow \boxed{E_n^{(1)} = 0}$$

the formula stated
in the problem.

Likewise, the second-order corrections are

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m^{(0)} | H_1 | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = F^2 \sum_{m \neq n} \frac{|\langle \psi_m^{(0)} | x | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

But we know that

$$|\langle \psi_m^{(0)} | x | \psi_n^{(0)} \rangle|^2 = \frac{\hbar}{2m\omega} \left[(n+1) \delta_{m, n+1} + n \delta_{m, n-1} \right].$$

Therefore, we obtain

$$E_n^{(2)} = \frac{F^2 \hbar}{2m\omega} \left(\frac{n+1}{-\hbar\omega} + \frac{n}{\hbar\omega} \right) = \frac{F^2}{2m\omega^2} (n - (n+1))$$

$$\Rightarrow \boxed{E_n^{(2)} = -\frac{F^2}{2m\omega^2}}$$

b) We write the Hamiltonian as

$$\begin{aligned} H &= H_0 + H_1 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 - Fx \\ &= \frac{p^2}{2m} + \frac{1}{2} m \omega^2 \left(x^2 - \frac{2Fx}{m\omega^2} \right) \\ &= \frac{p^2}{2m} + \frac{1}{2} m \omega^2 \left(x^2 - 2x \frac{F}{m\omega^2} + \frac{F^2}{m^2\omega^4} \right) - \frac{F^2}{2m\omega^2} \\ &= \frac{p^2}{2m} + \frac{1}{2} m \omega^2 \left(x - \frac{F}{m\omega^2} \right)^2 - \frac{F^2}{2m\omega^2} \end{aligned}$$

Introducing the new variables $p = \bar{p}$, $\bar{x} = x - \frac{F}{m\omega^2}$,
we obtain:

$$\bar{H} = \frac{\bar{p}^2}{2m} + \frac{1}{2} m \omega^2 \bar{x}^2 - \frac{F^2}{2m\omega^2}.$$

This problem has the (exact) 'eigen energies':

$$E_n^{\text{exact}} = \left(n + \frac{1}{2}\right) \hbar \omega - \frac{F^2}{2m\omega^2} = \underline{\underline{E_n^{(0)} + E_n^{(2)}}}$$

We see that the exact correction equals the second-order correction.

4) a) We require

$$1 \stackrel{!}{=} X^* X = |A|^2 (1 + 4 + 4) = 9 |A|^2 \Rightarrow A = 1/3$$

(we choose A to be real and positive)

b) Since we are dealing with a spin- $1/2$ system,
a measurement would obviously yield the value $\pm \hbar/2$.

Let us denote the spin-up state by $X_z^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and
the spin-down state by $X_z^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By definition,
the probability of finding the ~~state~~ system in
the state X_z^+ is:

$$|X_z^+ X|^2 = \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger \cdot \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2 = \frac{5}{9}; \text{ measured value} = \hbar/2$$

Likewise, we obtain the probability of finding
the system in the spin-down state:

$$|X_z^- X|^2 = \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\dagger \cdot \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2 = \frac{4}{9}; \text{ measured value} = -\hbar/2$$

$$\text{Also, } \langle S_z \rangle = \langle X | S_z | X \rangle = \hbar/18.$$

c) Using $\nabla_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we find the spin-up and spin-down
states w.r.t. the x -axis as:

$$X_x^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad X_x^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The measurements would again yield values $\pm \hbar/2$.
The measurements and probabilities now read:

$$+\hbar/2 : \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\dagger \cdot \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2 = \frac{13}{18}$$

$$-\hbar/2 : \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}^\dagger \cdot \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2 = \frac{5}{18}$$

$$\text{Also: } \langle S_x \rangle = \langle X | S_x | X \rangle = \frac{2}{9} \hbar$$

5) a) The energy E is determined by the kinetic energy alone:

$$\begin{aligned}
 \hat{H} \psi_{n_x n_y n_z} &= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) A \cdot \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \\
 &= \frac{\hbar^2}{2m} A \cdot \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2 + n_z^2}{L^2} \right) \pi^2 \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right) \\
 &= \underbrace{\frac{\hbar^2 \pi^2}{2m} \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2 + n_z^2}{L^2} \right)}_{= E} \cdot \psi_{n_x n_y n_z}
 \end{aligned}$$

$$\Rightarrow \boxed{E = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2 + n_z^2}{L^2} \right)}, \quad n_x, n_y, n_z \in \mathbb{N}.$$

We use the relation for the change in energy:

$$F_x \cdot dL_x = -dE$$

$$\Rightarrow F_x = - \frac{\partial E}{\partial L_x} = \frac{\hbar^2 \pi^2 n_x^2}{m L_x^3} = \frac{\hbar^2 \pi^2}{m L_x^3} \quad \square$$

for $n_x=1$ (ground state)

b) Eight identical spin- $1/2$ fermions can be described by the following quantum numbers :

$$(n_x, n_y, n_z) = (1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2).$$

Here, we assume that the system is in its ground state.

Note that each state (n_x, n_y, n_z) describes a one-particle state in which two fermions with opposite spin are found, for a total of $2 \times 4 = 8$ states.

The force depends only on the L_x -dependent contributions to the total energy:

$$E_{\text{tot}}^{(x)} = \frac{\hbar^2 \pi^2}{2m} \cdot \underset{\substack{\uparrow \\ \text{Spin up/down}}}{2} \cdot \left(\overset{n_x}{\begin{array}{c} \swarrow \quad \searrow \\ \frac{1}{L_x^2} + \frac{4}{L_x^2} + \frac{1}{L_x^2} + \frac{1}{L_x^2} \end{array}} \right) = 7 \frac{\hbar^2 \pi^2}{m L_x^2}.$$

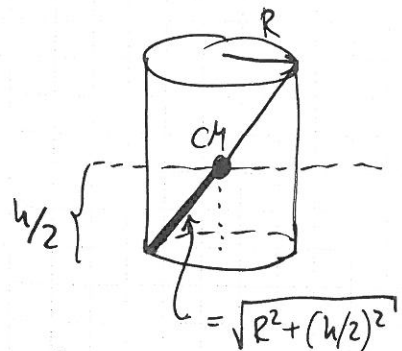
The force is then

$$F_x = - \left. \frac{\partial E_{\text{tot}}^{(x)}}{\partial L_x} \right|_L = - \left. \frac{\partial E_{\text{tot}}}{\partial L_x} \right|_L = 14 \frac{\hbar^2 \pi^2}{m L^3}.$$

6) First, we need to compute

$$\mathcal{F} = \frac{1}{h} \int_0^{x_0} |p(x)| dx = \frac{1}{h} \int_0^{x_0} \sqrt{2m(V-E)}$$

$$\begin{aligned} E=0 & \Rightarrow \frac{1}{h} \int_0^{x_0} \sqrt{2m mgx} \\ V=mgx & \end{aligned}$$



The can topples when the centre of mass (CM) is above the point of rotation, i.e. at maximum potential energy. At that moment, we have

$$x_0 = \sqrt{R^2 + (h/2)^2} - h/2, \quad (x_0 = 0.83 \text{ cm})$$

where x is the height of CM relative to its equilibrium value.

$$\Rightarrow \mathcal{F} = \frac{\sqrt{2m}}{h} \sqrt{mg} \int_0^{x_0} \sqrt{x} dx = \frac{m}{h} \sqrt{2g} \frac{2}{3} x^{3/2} \Big|_0^{x_0} = \frac{2m}{3h} \sqrt{2g} x_0^{3/2}$$

$$\Rightarrow \mathcal{F} \approx \frac{2 \cdot 0.3 \text{ Kg}}{3 \cdot 1.05 \cdot 10^{-34} \text{ Js}} \sqrt{2 \cdot 9.8 \text{ m/s}^2} \cdot (0.0083 \text{ m})^{3/2} = 6.4 \cdot 10^{30}$$

The lifetime is approximated by

$$\tau = \frac{2R}{V} e^{2\mathcal{F}}$$

The velocity equals the thermal velocity:

$$\frac{1}{2} m v^2 = \frac{1}{2} k_B T \Rightarrow v = \sqrt{\frac{k_B T}{m}}$$

Then, we finally obtain

$$\begin{aligned}\tau &= \frac{2 \cdot R}{v} e^{2\gamma} = 2R \cdot \sqrt{\frac{m}{k_B T}} e^{2\gamma} \\ &= 2 \cdot (0.03) \sqrt{\frac{0.3}{1.4 \cdot 10^{-23} \cdot 300}} e^{12.8 \cdot 10^{30}} \text{ s} \\ &= 5 \cdot 10^8 \cdot e^{12.8 \cdot 10^{30}} \text{ s} \\ &= 16 \cdot e^{12.8 \cdot 10^{30}} \text{ yrs.}\end{aligned}$$

This time vastly exceeds the age of the universe.