



QUM 1

Final Exam 2014

- Solutions -

1) a) Adding a_+ and a_- , we obtain

$$a_+ + a_- = \frac{1}{\sqrt{2}} 2 \left(\frac{m\omega}{\hbar} \right)^{1/2} x = \left(\frac{2m\omega}{\hbar} \right)^{1/2} x$$

$$\Rightarrow \boxed{x = \left(\frac{\hbar}{2m\omega} \right)^{1/2} (a_+ + a_-)}$$

b) We compute the expectation value as follows:

$$\begin{aligned} \langle 0 | x^4 | 0 \rangle &= \frac{\hbar^2}{4m^2\omega^2} \langle 0 | (a_+ + a_-)^4 | 0 \rangle \\ &= \frac{\hbar^2}{4m^2\omega^2} \langle 0 | a_+^4 + a_+^3 a_- + a_+^2 a_- a_+ + a_+^2 a_-^2 + a_+ a_- a_+^2 + a_+ a_- a_- a_+ \\ &\quad + a_+ a_-^2 a_+ + a_+ a_-^3 + a_- a_+^3 + a_- a_+^2 a_- + a_- a_+ a_- a_+ \\ &\quad + a_- a_+ a_-^2 + a_-^2 a_+^2 + a_-^2 a_+ a_- + a_-^3 a_+ + a_-^4 | 0 \rangle \end{aligned}$$

Now, we use $a_- |0\rangle = 0$ and $\langle 0 | a_+ = \langle 0 | a_-^\dagger = \langle a_- | = 0$:

$$\leadsto \langle 0 | x^4 | 0 \rangle = \frac{\hbar^2}{4m^2\omega^2} \langle 0 | a_- a_+^3 + a_- a_+ a_- a_+ + a_-^2 a_+^2 + a_-^3 a_+ | 0 \rangle$$

Next, we use $\langle 0 | a_- a_+^3 | 0 \rangle = 0$ and $\langle 0 | a_-^3 a_+ | 0 \rangle = 0$, see hint.

$$\Rightarrow \langle 0 | x^4 | 0 \rangle = \frac{\hbar^2}{4m^2\omega^2} \langle 0 | a_- a_+ a_- a_+ + a_-^2 a_+^2 | 0 \rangle$$

see hint $\hookrightarrow$

$$\begin{aligned} &= \frac{\hbar^2}{4m^2\omega^2} \left[\langle 0 | a_- a_+ a_- a_+ | 0 \rangle + \langle 0 | a_-^2 a_+^2 | 0 \rangle \right] \\ &= \frac{\hbar^2}{4m^2\omega^2} \left[\langle 0 | 0 \rangle + 2 \langle 0 | 0 \rangle \right] \end{aligned}$$

$$= \frac{3\hbar^2}{4m^2\omega^2} \quad \underline{\text{Answer:}} \quad \boxed{\langle 0 | x^4 | 0 \rangle = \frac{3\hbar^2}{4m^2\omega^2}}$$

- 2) a) i) For symmetry reasons, the probability p_x^{up} must be $p_x^{up} = 1/2$.
 ii) ——— " ———, $\langle S_x \rangle$ must be $\langle S_x \rangle = 0$.
 iii) After the measurement of the spin relative to the x -axis, it is either in the spin up or in the spin down state relative to the x -axis. Again for symmetry reasons, a subsequent measurement of the spin relative to the z -axis must yield $p_z^{up} = 1/2$.

Note: By "symmetry", we mean that space is assumed to be isotropic.

b) Let us denote

$\uparrow_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\downarrow_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as the spin eigenvectors of S_z , and $\uparrow_x = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\downarrow_x = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ as the spin eigenvectors of S_x . The latter can be seen easily by inspection. Then, we find:

$$i) p_x^{up} = \left[\left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]^2 = \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

$$ii) \langle S_x \rangle = (1 \ 0) S_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \ 0) \left[\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

iii) Either

$$p_z^{up} = \left[(1 \ 0) \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right]^2 = \frac{1}{2}$$

or

$$p_z^{up} = \left[(1 \ 0) \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right]^2 = \frac{1}{2}.$$

The answer is the same.

3) a) We write for the particle density ρ :

$$\rho = \frac{N}{V} = \frac{1}{2\pi^2} \int_0^{\infty} \frac{k^2}{e^{(\hbar^2 k^2 / 2m - \mu) / k_B T} - 1} dk, \quad \underline{\mu = 0.}$$

Now, we make the following substitutions:

$$\begin{aligned} x &= \frac{\hbar^2 k^2}{2m k_B T} \rightarrow k = \frac{\sqrt{2m k_B T}}{\hbar} \sqrt{x} \\ &\rightarrow dk = \frac{\sqrt{2m k_B T}}{2\hbar} \frac{1}{\sqrt{x}} dx \end{aligned}$$

This yields

$$\begin{aligned} \rho &= \frac{1}{2\pi^2} \left(\frac{2m k_B T}{\hbar^2} \right)^{3/2} \frac{1}{2} \int_0^{\infty} \frac{\sqrt{x}}{e^x - 1} dx \\ &= \frac{1}{4\pi^2} \left(\frac{2m k_B T}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{x^{\frac{3}{2}-1}}{e^x - 1} dx \\ \Rightarrow \rho &= \frac{1}{4\pi^2} \left(\frac{2m k_B T}{\hbar^2} \right)^{3/2} \cdot \Gamma\left(\frac{3}{2}\right) \cdot \zeta\left(\frac{3}{2}\right) \\ &= \frac{1}{4\pi^2} \left(\frac{2m k_B T}{\hbar^2} \right)^{3/2} \cdot \frac{\sqrt{\pi}}{2} \cdot 2.612 \\ &= 2.612 \cdot \left(\frac{m k_B T}{2\pi \hbar^2} \right)^{3/2} \end{aligned}$$

b) Solving the above expression for $T = T_c$ gives:

$$T_c = \frac{\hbar^2}{2m k_B} \cdot \left(\frac{4\pi^2 \rho}{\Gamma(\frac{3}{2}) \cdot \zeta(\frac{3}{2})} \right)^{2/3} \Rightarrow \boxed{T_c \sim \rho^{2/3}}$$

$$\left(T_c = \frac{2\pi \hbar^2}{m k_B} \cdot \left(\frac{\rho}{2.612} \right)^{2/3} \right)$$

4) a) The condition $V(x) = \infty$ for $x \leq 0$ requires that $\psi(0) = 0$. This means that for $x \geq 0$, the wave functions are simply described by those of the 1-D harmonic oscillator and $\psi(0) = 0$. Hence, the eigenfunctions are the "odd" ($n = 1, 3, 5, \dots$) eigenfunctions of the 1-D harmonic oscillator and the ground state ($n=1$) gives

$$E_0 = (n + \frac{1}{2}) \hbar \omega = \frac{3}{2} \hbar \omega.$$

b) The expectation value of the energy is given by

$$\langle H \rangle = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}. \text{ In what follows, the constant } c$$

will cancel out. We find

$$\langle H \rangle = \frac{\int_0^{\infty} dx x e^{-\alpha x} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 \right) x e^{-\alpha x}}{\int_0^{\infty} dx x^2 e^{-2\alpha x}}$$

In the first integral, we need to compute

$$\frac{d^2}{dx^2} (x e^{-\alpha x}) = (-2\alpha + \alpha^2 x) e^{-\alpha x}$$

$$\Rightarrow \langle H \rangle = \frac{\int_0^{\infty} dx e^{-2\alpha x} \left[\frac{\hbar^2}{2m} (2\alpha x - \alpha^2 x^2) + \frac{1}{2} m \omega^2 x^4 \right]}{\int_0^{\infty} dx x^2 e^{-2\alpha x}}$$

The integrals are of the form

$$\int_0^{\infty} dx x^n e^{-2\alpha x}.$$

They can be solved by use of a formula collection, integration by parts or parametric differentiation.

We obtain

$$\begin{aligned}\langle H \rangle &= \frac{\frac{\hbar^2}{2m} \left(2\alpha \frac{1!}{(2\alpha)^2} - \alpha^2 \frac{2!}{(2\alpha)^3} \right) + \frac{1}{2} m \omega^2 \frac{4!}{(2\alpha)^5}}{\frac{2!}{(2\alpha)^3}} \\ &= \frac{\hbar^2}{2m} \alpha^2 + \frac{3}{2} m \omega^2 \frac{1}{\alpha^2} \quad (*)\end{aligned}$$

To find the minimum, we differentiate:

$$\begin{aligned}0 &\stackrel{!}{=} \frac{d}{d\alpha} \langle H \rangle = \frac{\hbar^2}{m} \alpha - 3 m \omega^2 \frac{1}{\alpha^3} \\ \Rightarrow (\alpha^*)^4 &= 3 \frac{m^2 \omega^2}{\hbar^2}\end{aligned}$$

Substitution into (*) yields:

$$\langle H \rangle = \frac{\hbar^2}{2m} \sqrt{3} \frac{m \omega}{\hbar} + \frac{3}{2} m \omega^2 \frac{\hbar}{\sqrt{3} m \omega}$$

$$\Rightarrow \boxed{\langle H \rangle = \sqrt{3} \hbar \omega \approx 1.732 \hbar \omega > 1.5 \hbar \omega.}$$

The approximation is reasonable but not great.

5) ii) The Schrödinger equation (TE) reads

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x) \psi = E \psi.$$

Substituting $\psi(x) = A \cdot e^{\frac{i}{\hbar} S(x)}$, $A \neq 0$,

into it yields

$$\left(\psi' = A \frac{i}{\hbar} S' e^{\frac{i}{\hbar} S}, \quad \psi'' = \left(\frac{i}{\hbar} A S'' - \frac{A}{\hbar^2} (S')^2 \right) e^{\frac{i}{\hbar} S} \right) \quad \text{dash denotes derivative}$$

$$\boxed{-\frac{i\hbar}{2m} S'' + \frac{1}{2m} (S')^2 + V(x) - E = 0} \quad (*)$$

iii) Next, we expand $S(x)$ in a power series and substitute it into (*). We are only interested in order $\hbar^0$ and $\hbar^1$. Hence:

$$-\frac{i\hbar}{2m} (S_0'' + \hbar S_1'') + \frac{1}{2m} (S_0'^2 + 2\hbar S_0' S_1' + \hbar^2 S_1'^2) + V(x) - E = 0.$$

At each order, we find:

$$\hbar^0: \boxed{\frac{1}{2m} S_0'^2 + V(x) - E = 0} \quad (**), \quad \hbar^1: \boxed{S_0' S_1' - \frac{i}{2} S_0'' = 0} \quad (***)$$

iv) By inspection, we see that

$$S_0(x) = \pm \int^x p(x') dx'$$

and

$$S_1(x) = \frac{i}{2} \ln p(x)$$

are solutions.

$$(**): \frac{1}{2m} p^2(x) + V(x) - E = 0 \Rightarrow p(x) = \sqrt{2m(E - V(x))}. \checkmark$$

$$(***) : p(x) \cdot \frac{i}{2} \frac{p'(x)}{p(x)} - \frac{i}{2} p'(x) = 0. \checkmark$$



Finally, we find the approximate wave function to be

$$\psi(x) = A \cdot e^{\frac{i}{\hbar} S(x)} = A \cdot e^{\frac{i}{\hbar} (S_0 + \hbar S_1)}$$

$$= A \cdot e^{\frac{i}{\hbar} S_0(x)} \cdot e^{i S_1(x)}$$

$$= A \cdot e^{\pm \frac{i}{\hbar} \int^x p(x') dx'} \cdot e^{-\frac{1}{2} \ln p(x)}$$

$$= A \cdot e^{\ln(p(x))^{1/2}} \cdot e^{\pm \frac{i}{\hbar} \int^x p(x') dx'}$$

$$\Rightarrow \boxed{\psi(x) \approx \frac{A}{\sqrt{p(x)}} \cdot e^{\pm \frac{i}{\hbar} \int^x p(x') dx'}}$$

6) i) To find y_{n-1} , we simply replace Δx by $-\Delta x$:

$$y_{n-1} = y_n - \Delta x y'_n + \frac{(\Delta x)^2}{2} y''_n - \frac{(\Delta x)^3}{6} y^{(3)}_n + \frac{(\Delta x)^4}{24} y^{(4)}_n - \frac{(\Delta x)^5}{120} y^{(5)}_n$$

ii) Adding y_{n+1} and y_{n-1} gives

$$y_{n+1} + y_{n-1} = 2y_n + (\Delta x)^2 y''_n + \frac{(\Delta x)^4}{12} y^{(4)}_n \quad (*)$$

iii) In (*), we replace $y^{(4)}_n$ by

$$y^{(4)}_n = \frac{y''_{n+1} + y''_{n-1} - 2y''_n}{(\Delta x)^2}$$

and $y''_n = -K_n y_n$ etc.

$$\Rightarrow y^{(4)}_n = \frac{-K_{n+1} y_{n+1} - K_{n-1} y_{n-1} + 2K_n y_n}{(\Delta x)^2}$$

Substitution into (*) and rearranging yields the desired recursive equation for y_{n+1} in terms of y_n and y_{n-1} :

$$y_{n+1} + y_{n-1} = 2y_n + (\Delta x)^2 (-K_n y_n) + \frac{(\Delta x)^2}{12} (K_{n+1} y_{n+1} - K_{n-1} y_{n-1} + 2K_n y_n)$$

$$\Rightarrow \left(1 + \frac{(\Delta x)^2}{12} K_{n+1}\right) y_{n+1} = \left(2 - \frac{5}{6} (\Delta x)^2 K_n\right) y_n - \left(1 + \frac{(\Delta x)^2}{12} K_{n-1}\right) y_{n-1}$$

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