

Solutions

1) a) We demand that

$$1 \stackrel{!}{=} 2 |A|^2 \int_0^{\infty} e^{-2amx^2/\hbar} dx = 2 |A|^2 \frac{1}{2} \sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2 \sqrt{\frac{\pi \hbar}{2am}}$$

$$\Rightarrow A = \left(\frac{2am}{\pi \hbar} \right)^{1/4} \quad 3$$

b) We know that

$$\frac{\partial \psi}{\partial t} = -ia\psi, \quad \frac{\partial \psi}{\partial x} = -\frac{2amx}{\hbar} \psi, \quad \frac{\partial^2 \psi}{\partial x^2} = -\frac{2am}{\hbar} \left(\psi + x \frac{\partial \psi}{\partial x} \right) \\ = -\frac{2am}{\hbar} \left(1 - \frac{2amx^2}{\hbar} \right) \psi$$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \Leftrightarrow V\psi = i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

$$\Rightarrow V\psi = i\hbar(-ia)\psi + \frac{\hbar^2}{2m} \left(-\frac{2am}{\hbar} \right) \left(1 - \frac{2amx^2}{\hbar} \right) \psi$$

$$= \left[\hbar a - \hbar a \left(1 - \frac{2amx^2}{\hbar} \right) \right] \psi = 2a^2 m x^2 \psi \Rightarrow \boxed{V(x) = 2ma^2 x^2} \quad 3$$

c) We find

$$\langle x \rangle = \int_{-\infty}^{+\infty} x |\psi|^2 dx = 0 \quad 2, \quad \text{since integrand is odd.}$$

And so

$$\langle p \rangle = m \frac{d}{dt} \langle x \rangle = 0 \quad 2$$

2) First, we find the eigenvalues and eigenvectors of H :

$$\begin{vmatrix} a-E & 0 & b \\ 0 & c-E & 0 \\ b & 0 & a-E \end{vmatrix} = (a-E)(c-E)(a-E) - b^2(c-E)$$

$$= (c-E) \left[(a-E)^2 - b^2 \right] = 0 \Rightarrow \boxed{E_1 = c} \text{ or } \boxed{E_2 = a+b} \text{ or } \boxed{E_3 = a-b}$$

The eigenvectors follow from

$$\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = E_n \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

After some algebra, we find: (proof also by inspection/educated guess)

$$E_1 = c \hat{=} |s_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$E_2 = a+b \hat{=} |s_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$E_3 = a-b \hat{=} |s_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

We can now write

$$|s(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|s_2\rangle - |s_3\rangle),$$

and finally

$$\begin{aligned} |s(t)\rangle &= \frac{1}{\sqrt{2}} \left(e^{-iE_2 t/\hbar} |s_2\rangle - e^{-iE_3 t/\hbar} |s_3\rangle \right) \\ &= \frac{1}{\sqrt{2}} \left(e^{-i(a+b)t/\hbar} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - e^{-i(a-b)t/\hbar} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right) \end{aligned}$$

$$\Rightarrow |s(t)\rangle = e^{-iat/\hbar} \begin{pmatrix} -i \sin(bt/\hbar) \\ 0 \\ \cos(bt/\hbar) \end{pmatrix}$$

3) a) We compute

$$X^\dagger X = |A|^2 (9+16) = 25 |A|^2 \stackrel{!}{=} 1 \Rightarrow \boxed{A = 1/5}$$

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b) Straight forward calculation yields

$$\langle S_x \rangle = \frac{1}{25} \frac{\hbar}{2} (-3i \ 4) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-12i + 12i) = \boxed{0}$$

$$\langle S_y \rangle = X^\dagger S_y X = \frac{1}{25} \frac{\hbar}{2} (-3i \ 4) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-12 - 12) = \boxed{-\frac{12}{25} \hbar}$$

$$\langle S_z \rangle = X^\dagger S_z X = \frac{1}{25} \frac{\hbar}{2} (-3i \ 4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (9 - 16) = \boxed{-\frac{7}{50} \hbar}$$

c) We have $S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}.$$

Therefore, we find

$$(\sigma_{S_x})^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4} \Rightarrow \boxed{\sigma_{S_x} = \frac{\hbar}{2}}$$

Likewise, we compute

$$(\sigma_{S_y})^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25}\right)^2 \hbar^2 \Rightarrow \boxed{\sigma_{S_y} = \frac{7}{50} \hbar}$$

$$(\sigma_{S_z})^2 = \langle S_z^2 \rangle - \langle S_z \rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2 \hbar^2 \Rightarrow \boxed{\sigma_{S_z} = \frac{12}{25} \hbar}$$

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4) First, we determine the normalization constant

$$1 \stackrel{!}{=} 2 |A|^2 \int_0^{\infty} \frac{dx}{(x^2+b^2)^2} = 2 |A|^2 \frac{\pi}{4b^3} \Rightarrow \boxed{A = \sqrt{\frac{2b^3}{\pi}}} \quad 3$$

Next, we compute $\langle T \rangle$:

$$\begin{aligned} \langle T \rangle &= -\frac{\hbar^2}{2m} |A|^2 \int_{-\infty}^{+\infty} \frac{1}{(x^2+b^2)} \left(\frac{d^2}{dx^2} \frac{1}{x^2+b^2} \right) dx \\ &= -\frac{\hbar^2}{2m} |A|^2 \int_{-\infty}^{+\infty} \frac{1}{x^2+b^2} \frac{2(3x^2-b^2)}{(x^2+b^2)^3} dx \\ &= -\frac{\hbar^2}{2m} |A|^2 \int_{-\infty}^{+\infty} \left[2 \cdot \frac{3x^2}{(x^2+b^2)^4} - 2 \frac{b^2}{(x^2+b^2)^4} \right] dx \\ &= -\frac{4\hbar^2 b^3}{\pi m} \left[3 \int_0^{\infty} \frac{dx}{(x^2+b^2)^3} - 4b^2 \int_0^{\infty} \frac{dx}{(x^2+b^2)^4} \right] \\ \text{int. by parts} \quad &= -\frac{4\hbar^2 b^3}{\pi m} \left[3 \frac{3\pi}{16b^5} - 4b^2 \frac{5\pi}{32b^7} \right] = \boxed{\frac{\hbar^2}{4mb^2}} \quad 2 \end{aligned}$$

$$\text{likewise: } \langle V \rangle = \frac{1}{2} m \omega^2 |A|^2 2 \int_0^{\infty} \frac{x^2}{(x^2+b^2)^2} dx = m \omega^2 \frac{2b^3}{\pi} \frac{\pi}{4b} = \boxed{\frac{1}{2} m \omega^2 b^2} \quad 2$$

$$\Rightarrow \langle H \rangle = \frac{\hbar^2}{4mb^2} + \frac{1}{2} m \omega^2 b^2 \Rightarrow \frac{d\langle H \rangle}{db} \stackrel{!}{=} 0 \Rightarrow -\frac{\hbar^2}{2mb^3} + m \omega^2 b = 0$$

$$\Rightarrow b^4 = \frac{\hbar^2}{2m^2 \omega^2} \Rightarrow b^2 = \frac{1}{\sqrt{2}} \frac{\hbar}{m \omega}$$

$$\Rightarrow \langle H \rangle_{\min} = \frac{\hbar^2}{4m} \frac{\sqrt{2} m \omega}{\hbar} + \frac{1}{2} m \omega^2 \frac{1}{\sqrt{2}} \frac{\hbar}{m \omega} = \boxed{\frac{\hbar \omega}{\sqrt{2}}} \left(\Rightarrow \frac{1}{2} \hbar \omega \right) \checkmark$$

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5) a) By inspection, we find

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ eigenvalue } \boxed{V_0}; \quad X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ eigenvalue } \boxed{V_0};$$

$$X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ eigenvalue } \boxed{2V_0};$$

b) Characteristic equation yields:

$$\det(H - \lambda) = \begin{vmatrix} V_0(1-\epsilon) - \lambda & 0 & 0 \\ 0 & V_0 - \lambda & \epsilon V_0 \\ 0 & \epsilon V_0 & 2V_0 - \lambda \end{vmatrix} \stackrel{!}{=} 0$$

$$\Rightarrow [V_0(1-\epsilon) - \lambda] \cdot [(V_0 - \lambda)(2V_0 - \lambda) - (\epsilon V_0)^2] = 0$$

$$\Rightarrow \boxed{\lambda_1 = V_0(1-\epsilon)}$$

$$\text{and } (V_0 - \lambda)(2V_0 - \lambda) - (\epsilon V_0)^2 = 0$$

$$\Rightarrow \lambda^2 - 3V_0\lambda + (2V_0^2 - \epsilon^2 V_0^2) = 0$$

$$\Rightarrow \lambda = \frac{1}{2}(3V_0 \pm \sqrt{9V_0^2 - 4(2V_0^2 - \epsilon^2 V_0^2)}) = \frac{V_0}{2} [3 \pm \sqrt{1 + 4\epsilon^2}]$$

$$\approx \frac{V_0}{2} [3 \pm (1 + 2\epsilon^2)]$$

$$\Rightarrow \boxed{\lambda_2 = \frac{V_0}{2} (3 - \sqrt{1 + 4\epsilon^2}) \approx V_0(1 - \epsilon^2)}$$

$$\wedge \boxed{\lambda_3 = \frac{V_0}{2} (3 + \sqrt{1 + 4\epsilon^2}) \approx V_0(2 + \epsilon^2)}$$

c) We have

$$H' = \epsilon V_0 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow E_3' = \langle X_3 | H' | X_3 \rangle = \epsilon V_0 (0 \ 0 \ 1) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{0}. \quad |$$

($\leadsto$ no first-order correction)

The second-order correction reads:

$$E_3^2 = \sum_{m=1,2} \frac{|\langle X_m | H' | X_3 \rangle|^2}{E_3^0 - E_m^0}$$

But $\langle X_1 | H' | X_3 \rangle = 0$ (after some algebra)

and $\langle X_2 | H' | X_3 \rangle = \epsilon V_0 (0 \ 1 \ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \epsilon V_0$

and $E_3^0 - E_2^0 = 2V_0 - V_0 = V_0$

$$\Rightarrow E_3^2 = \frac{(\epsilon V_0)^2}{V_0} = \boxed{\epsilon^2 V_0} \quad 2 \text{ (second-order correction)}$$

$$\Rightarrow E_3 \approx E_3^0 + E_3' + E_3^2 = 2V_0 + 0 + \epsilon^2 V_0 = \boxed{V_0 (2 + \epsilon^2)}$$

(the same as we got for λ_3 in b).) |

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