

Eksamination in FY2450 Astrophysics

Wednesday June 8, 2016

Solutions

- 1a) Table 1 gives the spectral class and luminosity class of each of the 20 stars. The luminosity class of a star can (at least in principle) be read out of the Hertzsprung–Russell diagram, see Figure 1. Students were expected to get the luminosity classes approximately, but not exactly, right.

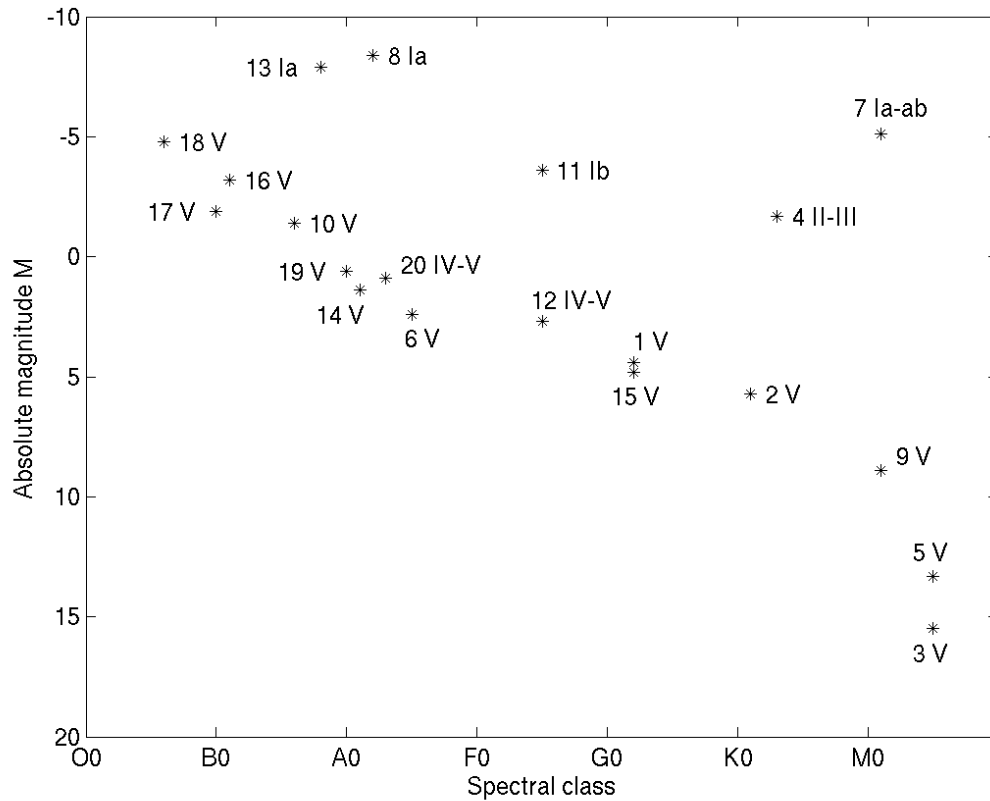
		Spectral class	Luminosity class
1	Alpha Centauri A	G2	V
2	Alpha Centauri B	K1	V
3	Proxima Centauri	M5	V
4	Alphard	K3	II-III
5	Barnard's star	M5	V
6	Beta Pictoris	A5	V
7	Betelgeuse	M1	Ia-ab
8	Deneb	A2	Ia
9	Kapteyn's star	M1	(V)
10	Merope	B6	Ve
11	Polaris	F7	Ib
12	Procyon	F5	IV-V
13	Rigel	B8	Ia
14	Sirius	A1	Vm
15	Sun	G2	V
16	Trapezium A	B1	V
17	Trapezium B	B0	V
18	Trapezium C	O6	V
19	Vega	A0	V
20	Zubenelgenubi	A4	IV-V

Tabell 1: Spectral class and luminosity class for 20 stars.

For our purposes we count Kapteyn's star as belonging to the main sequence, luminosity class V (no M star has had time to evolve beyond the main sequence), but it is actually classified as a subdwarf (main sequence stars are called dwarfs). It moves in a galactic orbit that takes it outside the disc of the Milky Way galaxy, in other words it belongs to the halo of the Milky Way. This means that it is very old and made of matter containing less of what astronomers call "metals", elements heavier than helium that are produced in stars of earlier generations. The difference in chemical composition makes it less luminous than standard main sequence stars of the same spectral class.

Merope, one of the Pleiades (seven sisters), is classified as Ve, the "e" telling that the spectrum has emission lines, believed to come from gas thrown out because the star rotates very fast.

Sirius is classified as Vm, because its spectrum shows stronger lines than usual of some metals.



Figur 1: The 20 stars plotted in a Hertzsprung–Russell diagram, with their luminosity classes shown. All the stars lying lowest in this diagram are main sequence stars, defined as luminosity class V. There is little difference between the luminosity classes in spectral classes O and B.

- 1b) The spectral class of a star tells its surface temperature, called effective temperature (defined as the temperature of a blackbody radiating the same energy flux as the surface of the star).

For a given temperature, the total radiation of a star is proportional to its radius squared, hence the radius determines the luminosity class.

A supergiant has a very large radius, hence a low surface gravity and a low density of its atmosphere. A low density gas produces narrow spectral lines. A higher density gas produces broader spectral lines, the lines are so called pressure broadened because of collisions between the atoms.

Another effect is that in a low density gas more atoms will be ionized, as described by the Saha equation. We may understand this intuitively as a result of a lower electron density: it takes a longer time for an ionized atom to recapture an electron.

- 2a) The square root expression describes the time dilatation: the slowing of a moving clock, which lowers the frequency of the emitted radiation.

The formula for the Doppler shift is expressed in terms of wave length and not in terms

of frequency, but the square root term has nothing to do with Lorentz contraction. In fact, the transverse Doppler effect changes the wave length of light perpendicular to the relative velocity of the light source and the observer, and there is no Lorentz contraction in this direction.

- 2b) This problem is an exercise in keeping the important terms and neglecting the unimportant. If, for example, v_r/c and v_t/c are of order 10^{-4} , then we neglect the second order terms $(v_r/c)^2$ and $(v_t/c)^2$, because they are of order 10^{-8} . But we can not neglect the first order terms, as some students do, because then we predict that there is no nonrelativistic Doppler effect at all, and that is wrong.

When we neglect the second order terms we set the square root equal to one. To first order we have that

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v_r}{c} ,$$

with no dependence on v_t . This nonrelativistic formula is so fundamental that students are expected to know it before they derive it here from the given relativistic formula.

The Earth moves around the Sun with speed

$$\frac{2\pi \text{ AU}}{1 \text{ year}} = \frac{2\pi \times 1.5 \times 10^8 \text{ km}}{365.25 \times 24 \times 3600 \text{ s}} = 29.87 \text{ km/s} = 1.0 \times 10^{-4} c .$$

The Doppler shift of the hydrogen alpha line due to this motion, when we move towards or away from a star that we observe, is

$$\Delta\lambda = \pm 10^{-4} \lambda_0 = 0.0656 \text{ nm} .$$

Many students have interpreted the question in a way which was not intended. They think only about the hydrogen alpha line in the spectrum of the Sun, and answer that it is Doppler shifted only because of the transverse Doppler effect, which is a second order effect, hence of order 10^{-8} . This must count as a correct answer.

- 2c) When $v_r = 0$ the two jets should have the same Doppler shift. In Figure 2 in the examination set we see that on day 114 the two Doppler shifted H_α lines are close together. The mid point between them is then roughly at $6750 \text{ \AA}$, which is presumably the transverse Doppler shift. In fact, on any day shown the mid point between the two moving lines is around $6750 \text{ \AA}$. The size of this Doppler shift from $6560 \text{ \AA}$ is $190 \text{ \AA}$. The equation to be solved for the velocity v is this:

$$\frac{6750}{6560} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

giving

$$\frac{v}{c} = \sqrt{1 - \left(\frac{656}{675}\right)^2} = \frac{\sqrt{675^2 - 656^2}}{675} = \frac{\sqrt{19 \times 1331}}{675} = 0.236 .$$

The jets move at 0.24 times the speed of light.

2d) We may rewrite the Doppler shift formula as

$$\frac{v_r}{c} = \frac{\lambda}{\lambda_0} \sqrt{1 - \frac{v^2}{c^2}} - 1 .$$

We already estimated the value of the square root to be $656/675$.

We read from the figure, on day 55, that the wave lengths for the two Doppler shifted lines are $5870 \text{ \AA}$ and $7660 \text{ \AA}$. Hence we should have

$$\frac{v_{\min}}{c} = \frac{587}{656} \times \frac{656}{675} - 1 = \frac{587}{675} - 1 = -\frac{88}{675} = -0.130 ,$$

and

$$\frac{v_{\max}}{c} = \frac{766}{656} \times \frac{656}{675} - 1 = \frac{766}{675} - 1 = \frac{91}{675} = 0.135 .$$

Fortunately we estimate the radial velocities to be smaller in absolute value than the total velocity $v = 0.24 c$. We also estimate $v_{\min} \approx -v_{\max}$, as expected.

2e) The Earth is slightly flattened because of its rotation. The tidal forces from the Moon and the Sun act on the equatorial bulge of the Earth to produce torques that cause the rotation axis of the Earth to precess.

2f) The kinetic energy of a mass m moving with velocity v is

$$E_k = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 = \frac{1}{2} mv^2 + \frac{3}{8c^2} mv^4 + \dots$$

where we have used the binomial formula

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots .$$

With $m = 1 \text{ kg}$ we get

$$mc^2 = 9 \times 10^{16} \text{ J} .$$

With $v/c = 0.01$ we get

$$E_k = \frac{1}{2} mv^2 = 5 \times 10^{-5} mc^2 = 4.5 \times 10^{12} \text{ J} .$$

With $v/c = 0.5$ the exact relativistic formula gives

$$E_k = \frac{mc^2}{\sqrt{1 - 0.5^2}} - mc^2 = \frac{mc^2}{\sqrt{0.75}} - mc^2 = 0.1547 mc^2 = 1.39 \times 10^{16} \text{ J} ,$$

but the non-relativistic formula is still not very bad,

$$E_k = \frac{1}{2} mv^2 = \frac{1}{8} mc^2 = 0.125 mc^2 = 1.125 \times 10^{16} \text{ J} .$$

For comparison,

$$4 \text{ Twh} = 4 \times 10^{12} \times 3600 \text{ J} = 1.44 \times 10^{16} \text{ J} .$$

To accelerate 1 kg to half the speed of light takes all the electric energy consumed in Trondheim for one year.

Many students did not know the relativistic formula for kinetic energy. They were not punished for this, because they could use the nonrelativistic formula and get acceptable answers.

A more fundamental problem is that some students are confused about the concepts of mass and rest mass. It is unfortunate that physicists at some time introduced a concept of velocity dependent relativistic mass. In FY2450 mass always means rest mass.

The confusion leads many to invent the following wrong formula for the kinetic energy:

$$E_k = \frac{1}{2} \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

They simply take the nonrelativistic formula and replace the rest mass by the velocity dependent relativistic mass.

- 3a) An absorption line is produced when the conditions are such that a photon can excite an atom or a molecule from a lower to a higher level. The axis in the plot showing the spectral classes, OBAFGKM, is a reversed temperature axis. With increasing temperature, when we go from right to left in the plot, a given spectral line first appears when the temperature becomes high enough that photons get enough energy to excite the atoms (or molecules). The line is strongest when there are many atoms to excite and many photons to excite them. The line disappears again with increasing temperature, because more and more of the atoms become excited, so that finally no atoms remain in the lower level. Atoms can bind into molecules only in the atmospheres of the coolest stars.

The higher the excitation energy of the atomic transition, the higher the temperature where the spectral line appears. For example, it takes less energy to remove the first electron from a helium atom than it takes to excite the second electron afterwards. Therefore the lines of neutral helium appear before those of ionized helium.

Many students claim that what we see on the surface of the star is directly influenced by fusion processes producing new elements. Such ideas are wrong. Except for white dwarfs and neutron stars, it is only deep inside a star that temperatures and densities are high enough for fusion processes to take place. New elements are produced deep down, afterwards they usually stay there and are not seen on the surface. In particular, few stars burn more than a fraction of their hydrogen. Exceptions are the red dwarf stars, of class M, where the matter throughout the star is mixed by convection. On the other hand, at the present age of the Universe, no red dwarf has had time to burn more than a fraction of its hydrogen.

- 3b) We build big telescopes to collect much light, and to get high angular resolution. The amount of light they collect per time is proportional to the area of their light opening.

And the smallest angle they can resolve (in the absence of atmospheric turbulence) is inversely proportional to the diameter of the opening.

A misunderstanding is that long telescopes have better angular resolution. It is true that a long telescope produces a large image for photography, but the angular resolution of a telescope is limited by the diffraction of light. We reduce diffraction by increasing the diameter of the light opening.

We launch some big telescopes into space in order to avoid the atmospheric turbulence, this improves the angular resolution further. The second advantage of telescopes above the atmosphere is that they can observe radiation that does not reach the telescopes on the ground. The atmosphere absorbs completely the electromagnetic radiation in large parts of the spectrum: gamma rays, X rays, ultraviolet, partly also infrared and radio waves.

- 3c) The visible magnitude of Proxima Centauri at distance $d = 1.3$ parsec is

$$m_V = M_V + 5 \log_{10} \left(\frac{d}{10 \text{ parsec}} \right) = 15.5 + 5 \log_{10} 0.13 = 15.5 - 4.43 = 11.07 .$$

This is five magnitudes, or 100 times, fainter than what we can see without a telescope.

If Proxima Centauri could replace Jupiter, its distance would vary between 4.2 AU and 6.2 AU. One parsec is 206 265 AU. Hence its brightness would vary between

$$m_V = M_V + 5 \log_{10} \left(\frac{4.2 \text{ AU}}{2062650 \text{ AU}} \right) = 15.5 - 28.46 = -12.96 .$$

and

$$m_V = M_V + 5 \log_{10} \left(\frac{6.2 \text{ AU}}{2062650 \text{ AU}} \right) = 15.5 - 27.61 = -12.11 .$$

It would shine like the full Moon, but look rather pointlike, having an angular diameter only 40 % larger than Jupiter.

- 3d) The luminosity of a star is proportional to the square of its radius and the fourth power of its surface temperature. Hence the luminosity of Proxima Centauri relative to the Sun is

$$\frac{L}{L_{\odot}} = 0.141^2 \times \left(\frac{3042}{5780} \right)^4 = 1.525 \cdot 10^{-3} .$$

Since a difference of 5 magnitudes is a factor of 100 in luminosity, a difference of 2.5 magnitudes is a factor of 10 in luminosity, and a factor of $1.525 \cdot 10^{-3}$ in luminosity is a difference in magnitudes of

$$-2.5 \log_{10}(1.525 \cdot 10^{-3}) = 7.04 .$$

Remember the convention that a fainter star has a higher magnitude.

The difference in absolute visual magnitude is not 7.04 but $15.5 - 4.8 = 10.7$. This is a factor in luminosity of

$$\frac{L}{L_{\odot}} = 10^{-\frac{10.7}{2.5}} = 5.25 \cdot 10^{-5} .$$

There is an apparent inconsistency here. But we should remember that the difference in magnitude of 7.04 includes all the electromagnetic radiation of the two stars, whereas the difference in visual magnitude includes only the radiation in the visible part of the spectrum. Proxima Centauri is so cold that about 90 % of its radiation is in the infrared. For comparison, about 50 % of the radiation of the Sun is in the infrared.

Hence the two numbers 7.04 and 10.7 may be consistent. This is not a very precise consistency check, but students at the examination could not be expected to do more.

We may try to do better here. The standard V filter admits light with wave lengths around 550 nm, in the middle of the visible part of the spectrum, actually between $\lambda_1 = 500$ nm and $\lambda_2 = 600$ nm. The fraction of blackbody light at temperature T having wave lengths in this interval is

$$f = \frac{\int_{x_2}^{x_1} dx \frac{x^3}{e^x - 1}}{\int_0^\infty dx \frac{x^3}{e^x - 1}},$$

where $x_1 = hc/(kT\lambda_1)$ and $x_2 = hc/(kT\lambda_2)$. For the Sun, at $T = 5780$ K, we compute $f_\odot = 0.12729$, and for Proxima Centauri, at $T = 3040$ K, we compute $f = 0.02717$. The ratio $f/f_\odot$ converted to magnitudes is

$$\Delta m = -2.5 \log_{10} \frac{f}{f_\odot} = 1.68.$$

This is less than half the difference between 10.7 and 7.0. Some inconsistency remains, unfortunately.

- 3e) Using the assumption that the central temperature is proportional to M/R , we compute that Proxima Centauri should have a central temperature of

$$T_c = \frac{0.123}{0.141} \times 15.7 \times 10^6 \text{ K} = 13.7 \times 10^6 \text{ K}.$$

This is high enough that the fusion of hydrogen to helium can take place. Especially since the fusion rate is proportional to the density squared, and the density of a star scales as M/R^3 , so that the density of Proxima Centauri is higher than the density of the Sun by a factor

$$\frac{0.123}{0.141^3} = 43.9.$$

The same estimate for Jupiter (very suspicious in this case) says that its central temperature should be

$$T_c = \frac{0.0010}{0.10} \times 15.7 \times 10^6 \text{ K} = 1.57 \times 10^5 \text{ K}.$$

This can not be taken seriously as a precise quantitative estimate, but it seems safe to conclude that the central temperature of Jupiter is too low for hydrogen fusion. Especially since the average density of Jupiter is equal to the average density of the Sun (M/R^3 for Jupiter equals M/R^3 for the Sun).