



Exam in FY3106/FY8303 Phase Transitions and Critical Phenomena

Saturday, June 11, 2016

09:00–13:00

Allowed help: Alternativ C

This problem set consists of 2 pages, plus an Appendix of one page.

Problem 1

As a result of a renormalization group transformation, the free energy has the form

$$\omega(\tau, h) = \omega_0 + \frac{1}{l^d} \omega_s(l^{y_\tau} \tau, l^{y_h} h) , \quad (1)$$

where $\tau = (T - T_c)/T_c$ is the reduced temperature, h is the magnetic field and l the scale factor.

- a) Use this expression to derive the heat capacity exponent α in terms of y_τ .
- b) Use the renormalization group to derive the hyperscaling relation $2 - \alpha = d\nu$.

Problem 2

A spin system belonging to the universality class $(d, n) = (2, 2)$ consists of classical spins of length $s = 1$ on a square lattice. The orientation of spin i is defined by its angle ϕ_i with the x -axis. The hamiltonian in zero external magnetic field is

$$H = -J \sum_{\langle i, j \rangle} \vec{s}_i \cdot \vec{s}_j = -J \sum_{\langle i, j \rangle} \cos(\phi_i - \phi_j) , \quad (2)$$

where the sum runs over nearest neighbors. The spin-spin correlation function is defined as

$$\Gamma(\vec{r}) = \langle \vec{s}(\vec{r}) \cdot \vec{s}(\vec{0}) \rangle . \quad (3)$$

- a) Show that at sufficiently low temperature, the correlation function may be written

$$\Gamma(\vec{r}) = \exp \left[\langle [\phi(\vec{r}) - \phi(\vec{0})] \phi(\vec{0}) \rangle \right] . \quad (4)$$

- b) For large $r = |\vec{r}|$ we have in two dimensions $\Gamma(\vec{r}) \sim r^{-\eta}$. In the gaussian approximation, we furthermore have that

$$\langle \phi(\vec{r}) \phi(\vec{0}) \rangle \sim \text{Const} - \frac{k_b T}{2\pi J} \ln \left(\frac{r}{r_0} \right) . \quad (5)$$

Use this to determine η .

- c) Why is this result for η not valid for all temperatures? Which mechanism destroys the result?

Problem 3

We consider the Ising model on a triangular lattice using the Niemeijer-van Leeuwen 3-spin block renormalization group construction with the majority rule. We only consider the first cumulant approximation.

- a) Let the initial hamiltonian only contain the nearest neighbor coupling K_1 (with $1/k_B T$ absorbed into it). Show that the renormalization group transformation becomes

$$K'_1 = 2\psi^2(K_1)K_1, \quad (6)$$

where

$$\psi(K_1) = \frac{e^{3K_1} + e^{-K_1}}{e^{3K_1} + 3e^{-K_1}}. \quad (7)$$

- b) Show that one of the fixed points in this transformation is

$$K_1^* = \frac{1}{4} \ln[1 + 2\sqrt{2}]. \quad (8)$$

Determine the two other fixed points. Which of the fixed points describe the critical point in the model? What is the stability of the fixed points?

- c) Determine the exponent y_τ defined in equation (1) using the renormalization group transformation, equation (6), explaining the steps on the way. We give as information that

$$\left(\frac{dK'_1}{dK_1} \right)_{K_1=K_1^*} \approx 1.62352... , \quad (9)$$

where K_1^* is the fixed point given in equation (8).

The following information may be of some use:

Heat capacity near a critical point: $c = \partial^2 \omega / \partial \tau^2 \sim |\tau|^{-\alpha}$.

Correlation length near a critical point: $\xi \sim |\tau|^{-\nu}$.

Cumulant expansion: $\langle e^{\lambda V} \rangle = \exp [\sum_{n=1}^{\infty} \lambda^n C_n]$ where C_n is the n th cumulant. $C_1 = \langle V \rangle$, $C_2 = (\langle V^2 \rangle - \langle V \rangle^2)/2$ etc.