

Institutt for fysikk, NTNU

Takehome Exam FY3106/FY8303 Phase transitions and critical phenomena
12:00 December 2- 12:00 December 9, 2019

Problem 1

The Landau (mean-field) energy functional of phase transitions, is given by

$$E[\phi] = \frac{r}{2}\phi^2 + u\phi^4 + v\phi^6 - h\phi$$

where h is an external uniform magnetic field, and ϕ is an order parameter field which we take here to be scalar and real-valued.

- a) Show that when $h = 0$, this theory predicts a continuous phase transition when $u > 0$ and a discontinuous phase transition when $u < 0$.
- b) Find the value of $r = r(u, v)$ where the transitions occur.
- c) Set $v = 1$ and sketch a phase-diagram in (r, u) -space.

Problem 2

The Ginzburg-Landau energy functional for an n -component ferromagnet, is given by (in the standard notation used in lectures for r and u)

$$E[\mathbf{m}] = \int d^d r \left(\frac{1}{2} (\nabla \mathbf{m})^2 + \frac{r}{2} \mathbf{m}^2 + u (\mathbf{m}^2)^2 + v \sum_{\alpha=1}^n m_{\alpha}^4 \right)$$

where $\mathbf{m}$ is an n -component vector, $\mathbf{m}^2 = \sum_{i=1}^n m_i m_i$, and $(u > 0, v \geq 0)$.

- a) Find, to one-loop order, the recursion relations on differential form for (r, u, v) .
- b) Find all the fixed points of the recursion relations.
- c) Assume that the system is at its critical temperature, $r = r_c(u, v)$. Investigate the flows in (u, v) -space and find the stable fixed points for $n < n_c$ and $n > n_c$, where $n_c = 4 + \mathcal{O}(\epsilon)$. Here, $\epsilon = 4 - d$.
- d) Find the temperature exponent for each of these fixed points and deduce which will be observed experimentally.

Problem 3

In lectures, we have considered the RG equations for the overall charge-neutral $2D$ Coulomb gas, and found the following recursion relations for the fugacity $\zeta(l)$ and temperature $\tau(l)$

$$\begin{aligned} \frac{d\tau}{dl} &= \frac{1}{2} \zeta^2 \\ \frac{d\zeta^2}{dl} &= \left(4 - \frac{1}{\tau} \right) \zeta^2 \end{aligned}$$

These recursion relations describe a low-temperature insulating phase and a high-temperature metallic phase separated by the separatrix $\zeta = -|T(l)|$, where $T(l) \approx 4\tau(l) - 1$.

a) Consider a d -dimensional overall charge-neutral system of N_+ positive point-charges $q_i = +1$ and N_- negative point-charges with charge $q_i = -1$, interacting with the potential

$$V(r) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{(4\pi)^{d/2}} \left[\left(\frac{r}{a}\right)^{2-d} - 1 \right]$$

Derive the recursion relations for $(\zeta(l), T(l))$, draw the flow diagram, and find the possible phases of this system, when $d \geq 3$. Here, a is a short-distance cutoff.

b) Consider a lattice gauge-theory in $d = 3$ dimensions on a simple cubic lattice, with the partition function

$$\begin{aligned} Z &= \int_0^{2\pi} \prod_i d\mathbf{A}_i e^{\kappa \sum_{i,\mu} \cos(B_{i,\mu})} \\ B_{i,\mu} &= (\boldsymbol{\Delta} \times \mathbf{A})_{i,\mu} \end{aligned}$$

where Δ_μ denotes a lattice gradient in direction μ , $\Delta_\mu f_i = f_{i+\hat{\mu}} - f_i$, $\boldsymbol{\Delta} \times \mathbf{A}$ is a lattice-curl, and $\hat{\mu}$ is a unit vector in direction μ . Use the Villain-approximation and show that the partition function may be written on the form

$$Z = \sum_{\{b_{i,\mu}\}} e^{-\frac{1}{2\kappa} \sum_{i,\mu} b_{i,\mu}^2}$$

with the constraint

$$\boldsymbol{\Delta} \times \mathbf{b} = 0$$

on each elementary plaquette on the lattice.

c) Solve the constraint and show that the partition function may be expressed as

$$Z = \sum_{\{l_i\}} e^{-\frac{1}{2\kappa} \sum_{i,\mu} (l_{i+\mu} - l_i)^2}$$

where $\{l_i\}$ are integers defined on the dual lattice.

d) Use the Poisson-summation formula, elevate l_i to real-valued fields θ_i , and integrate out the $\{\theta_i\}$ -fields. From this, deduce the phase(s) of the lattice gauge-theory given above.