1

Final exam TFY 3106 / FY 8303

Problem 1

a) Singular part of heat capacity:

$$C_s = \frac{1}{\ell^d}\frac{\partial^2 w_s}{\partial \tau^2} \sim |\tau|^{-\alpha}.$$

$\Longrightarrow$

$$\frac{1}{\ell^d} w_s \sim \frac{1}{\ell^d} |\ell^{y_\tau} \tau|^{2-\alpha}$$

$$= \ell^{-d + y_\tau(2-\alpha)} |\tau|^{2-\alpha} \sim |\tau|^{2-\alpha}$$

This expression cannot depend on ℓ

$\Rightarrow$

$$-d + y_\tau (2-\alpha) = 0 \Longrightarrow$$

$$\underline{\underline{\alpha = 2 - \frac{d}{y_\tau}}}$$

b) $\xi(\ell^{y_\tau}\tau) = \frac{1}{\ell}\xi(\tau) \Rightarrow$

$$|\ell^{y_\tau}\tau|^{-\nu} \sim \ell^{-1} |\tau|^{-\nu}$$

$$\Rightarrow \quad \underline{\underline{\nu = \frac{1}{y_\tau}}}$$

Combining this expression with

$$\left.\begin{aligned} 2-\alpha &= \frac{d}{y_t} \\ \frac{1}{y_t} &= \nu \end{aligned}\right\} \Rightarrow \underline{\underline{2-\alpha = d\nu}}$$

Problem 2

a) $\underline{T(\vec{r})} = \langle \cos(\phi(\vec{r}) - \phi(\vec{0}))\rangle$

$$= \langle \cos(\phi(\vec{r}) - \phi(\vec{0}))\rangle + i\underbrace{\langle \sin(\phi(\vec{r}) - \phi(\vec{0}))\rangle}_{=0}$$

Since

$$\langle \sin(\phi(\vec{r}) - \phi(\vec{0}))\rangle$$

$$= \langle \sin(\phi(\vec{0}) - \phi(\vec{r}))\rangle$$

$$= -\langle \sin(\phi(\vec{r}) - \phi(\vec{0}))\rangle$$

$$\underline{= \langle e^{i(\phi(\vec{r}) - \phi(\vec{0}))}\rangle}$$

Cumulant expansion:

$$\langle e^{i(\phi(\vec{r}) - \phi(\vec{0}))}\rangle = e^{-\frac{1}{2}\langle(\phi(\vec{r}) - \phi(0))^2\rangle + \overbrace{\dots}^{}}$$

Drop at low temperature.

We have

$$\underline{\langle(\phi(\vec{r})-\phi(\vec{0}))^2\rangle} = \langle\phi(\vec{r})^2\rangle - 2\langle\phi(\vec{r})\phi(\vec{0})\rangle$$

$$+\langle\phi(\vec{0})^2\rangle = *$$

We use translational symmetry:

$$\langle\phi(\vec{r})^2\rangle = \langle\phi(\vec{0})^2\rangle :$$

$$* = \langle\phi(\vec{0})^2\rangle - 2\langle\phi(\vec{r})\,\phi(\vec{0})\rangle$$

$$+\langle\phi(\vec{0})^2\rangle$$

$$= 2\langle\phi(\vec{0})^2\rangle - 2\langle\phi(\vec{r})\phi(\vec{0})\rangle$$

$$= \underline{-2\langle(\phi(\vec{r})-\phi(\vec{0}))\phi(0)\rangle}$$

- And we have the final answer:

$$\underline{\underline{\Gamma(\vec{r}) = e^{\langle(\phi(\vec{r})-\phi(\vec{0}))\phi(\vec{0})\rangle}}}$$

b) $$\Gamma(r) = e^{\langle(\phi(\vec{r})-\phi(\vec{0}))\phi(\vec{0})\rangle} = e^{-\langle\phi(\vec{0})^2\rangle}$$

$$\cdot\, e^{\langle\phi(\vec{r})\phi(\vec{0})\rangle}$$

4

$$= e^{-\langle \phi(\vec{0})^2 \rangle} \, e^{+\text{Const} - \frac{k_B T}{2\pi J} \ln\left(\frac{r}{r_0}\right)}$$

$$= e^{\text{Const} - \langle \phi(\vec{0})^2 \rangle} \left(\frac{r}{r_0}\right)^{-\frac{k_B T}{2\pi J}} \sim r^{-\eta}$$

$$\Rightarrow \quad \underline{\underline{\eta = \frac{k_B T}{2\pi J}}}$$

c) At high temperature, the spins will be decorrelated and $\Gamma(\vec{r})$ must fall off exponentially. The result in b) shows that the correlation function falls off algebraically. Hence, there must be a temperature at which the behavior switches from algebraic to exponential. This is where the <u>Kosterlitz - Thouless transition</u> occurs at which <u>vortex-antivortex pairs appear</u>. These destroy the algebraic behavior.

Problem 3

a)

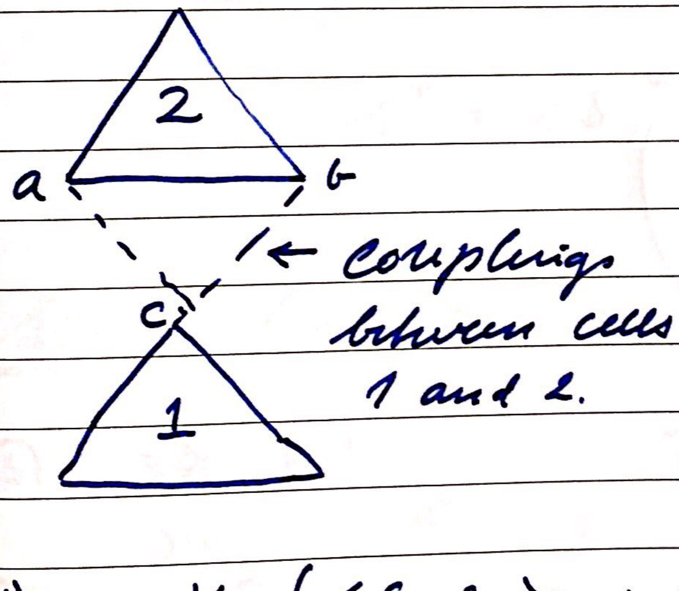
$$H(s) = H^{\circ}(s) + V(s)$$

intracell couplings   Intercell couplings.

$$e^{\Omega_0 + H'(s')} = e^{N' \ln Z_0 + \langle V \rangle_{s'}}$$

where

$$\langle V \rangle_{s'} = \frac{\sum'_{s} P(s',s)\, e^{H^{\circ}(s)}\, V(s)}{\sum'_{s} P(s',s)\, e^{H^{\circ}(s)}}$$



$$\langle V \rangle_{s'} = K_1 \left(\langle s_a s_c \rangle_{s'} + \langle s_b s_c \rangle_{s'} \right)$$

$$= K_1 \langle s_c \rangle_{s'} \left(\langle s_a \rangle_{s'} + \langle s_b \rangle_{s'} \right)$$

$$\langle s_c \rangle_{s'} = \frac{e^{3K_1} + e^{-K_1} + e^{-K_1} - e^{-K_1}}{e^{3K_1} + e^{-K_1} + e^{-K_1} + e^{-K_1}}\, s'$$

$$\begin{matrix} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \uparrow\uparrow \\ \uparrow & \uparrow & \uparrow & \downarrow \end{matrix}$$

$$= s' \frac{e^{3K_1} + e^{-K_1}}{e^{3K_1} + 3e^{-K_1}} = s'\varphi(K_1)$$

$$\langle V \rangle_{s'} = \underbrace{2K_1 \varphi^2(K_1)}_{K_1'} s_1' s_2'$$

$$\Rightarrow \quad \underline{\underline{K_1' = 2K_1 \varphi^2(K_1)}}$$

b) Fixed points:

$$K_1^* = 2K_1^* \varphi^2(K_1^*)$$

<u>Stable</u> (attractive) fixed points:

$$K_1^* = \begin{cases} 0 \\ \infty. \end{cases}$$

<u>Unstable</u> fixed point:

Describe the critical properties.

$$1 = 2\varphi^2(K_1^*) \quad \Rightarrow \quad \underline{\underline{K_1^* = \tfrac{1}{4}\ln(1+2\sqrt{2})}}$$

c) Near K_1^*, we set $K_1 = K_1^* + u_\tau$.

R.G. transformation:

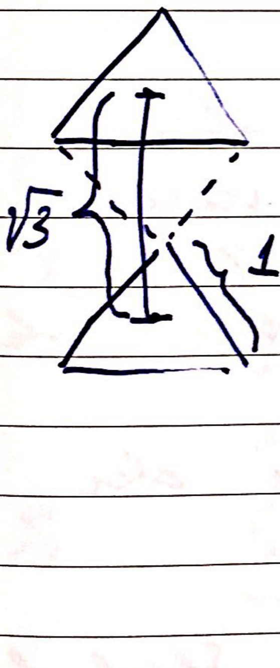
$$K_1' = K_1'(K_1^* + u_\tau) = \overset{K_1^*}{\overset{\shortparallel}{K_1'(K_1^*)}}$$

$$+ \left(\frac{dK_1'}{dK_1}\right)_{K_1^*} u_\tau$$

$$= K_1^* + \left(\frac{dK_1'}{dK_1}\right)_{K_1^*} u_\tau$$

We also define

$$K_1' = K_1^* + u_\tau' \longrightarrow u_\tau' = \left(\frac{dK_1'}{dK_1}\right)_{K_*'} u_\tau$$



Rescaling factor : $l = \sqrt{3}$

$$u_\tau' = \left(\frac{dK_1'}{dK_1}\right)_{K_*'} u_\tau$$

$$= l^{y_\tau} u_\tau \Rightarrow$$

$$y_\tau = \frac{\ln \left(\frac{dK_1'}{dK_1}\right)_{K_*'}}{\ln \sqrt{3}} \simeq 0.882203\ldots$$