Exam FY3106/FY8303

December 1, 2023

1a)

$$Z = \int D\theta \; e^{-H}$$

H must therefore be dimensionless

$$H = \frac{\rho}{2} \int d^d r \, (\nabla\theta)^2$$

θ must be dimensionless:

$$\psi = |\psi| e^{i\theta}$$

$$[H] = L^0 = [\rho] \; L^{d-2}$$

$$\underline{\underline{[\rho] = L^{2-d}}}$$

b) From the formula-sheet, we see that $\gamma \sim \rho \sim |\psi|^2$ (density)

$$|\langle\psi\rangle| \sim |t|^\beta$$

Square this to get a density $\Rightarrow \sim |t|^{2\beta}$

$$\gamma \sim |t|^{\omega}$$

$|\langle\psi\rangle|^2$ and Υ both the dimension $\Rightarrow$ natural to compare ω to 2β

c) $$\Upsilon \sim \rho \sim L^{2-d} \sim |t|^{\nu(d-2)}$$

$$\omega = \nu(d-2)$$

$$2\beta - \omega = 2\beta - \nu d + 2\nu$$

We are asked to express $2\beta-\omega$ in terms of critical exponents only, so we must eliminate νd

$$\alpha = 2 - d\nu \Rightarrow d\nu = 2-\alpha$$

$$2\beta - \omega = 2\beta + \alpha - 2 + 2\nu$$

This can be simplified using

$$\alpha + 2\beta + \gamma = 2 \Rightarrow 2\beta + \alpha - 2 = -\gamma$$

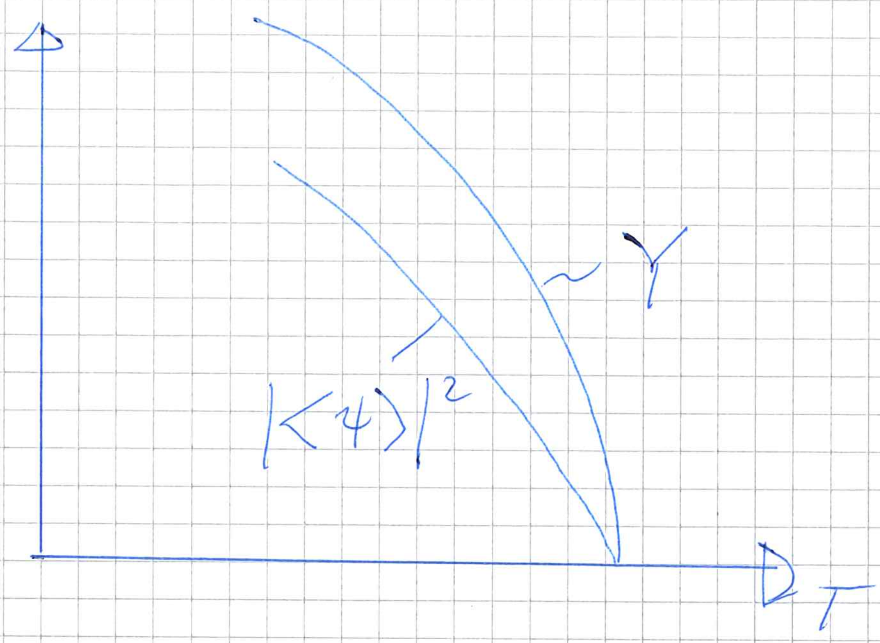
$$2\beta - \omega = 2\nu - \gamma = 2\nu - \nu(2-\eta)$$
$$= \eta\nu$$

$$\eta, \nu > 0 \Rightarrow$$

$$2\beta > \omega$$

This means that Υ approaches 0 as $T \to T_c$ with a steeper slope than $|\langle\psi\rangle|^2$

d)



e) 3D XY-model, spin-waves

$$\left\langle e^{i(\theta(\vec r)-\theta(\vec r'))}\right\rangle$$

$$= e^{-I(r)}$$ following steps exactly as in lecture notes.

$$I(r) = \frac{1}{K}\frac{1}{(2\pi)^3}\int d^3q \left(\frac{1-\cos(\vec q\cdot\vec r)}{q^2}\right)$$

Instead of doing a 2D $\vec q$-integral, we are doing a 3D integral.

Note that when $\vec q \to 0$, the integral is less divergent in $D=3$ than in $D=2$.

Using the formula on the exam-paper, we have

$$I(r) = \frac{1}{K} \frac{1}{(2\pi)^2} \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dq \frac{q^2 (1 - \cos(qr\cos\theta))}{q^2}$$

The θ-integral of the first term gives $\int_0^{\pi} d\theta \sin\theta = - \Big|_0^{\pi} \cos\theta = \underline{2}$

Next:

$$\int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dq \cos(qr\cos\theta)$$

This is the real part of

$$\int_0^{\infty} dq \int_0^{\pi} d\theta \sin\theta \, e^{iqr\cos\theta}$$

Consider first the θ-integral:

$x = \cos\theta$

$dx = -\sin\theta \, d\theta$

$$-\int_1^{-1} dx \, e^{iqrx} = \int_{-1}^{1} dx \, e^{iqrx}$$

$$= \frac{1}{iqr} \left(e^{iqr} - e^{-iqr} \right) = \frac{2}{qr} \sin(qr)$$

We are then left with the integral

$$\int_0^\infty dq \, \frac{2}{qr} \, \sin(qr)$$

$$= \frac{2}{r} \int_0^\infty du \, \frac{\sin u}{u} = \frac{2}{r} \frac{\pi}{2}$$

$$I(r) = \frac{1}{K} \frac{1}{(2\pi)^2} \left(2 - \frac{\pi}{2} \frac{2}{r} \right)$$

$$\langle e^{i\theta(r)} e^{-i\theta(0)} \rangle = e^{-I(r)}$$

$$= e^{-\frac{1}{2\pi^2 K}} \, e^{\frac{1}{4\pi K} \frac{1}{r}} \equiv G(r)$$

$$\frac{1}{r} \xrightarrow{r \to \infty} 0$$

Thus, the correlation function decays as a function of r, as expected, but it approaches a non-zero value as $r \to \infty$.

$$\lim_{r\to\infty} G(r) = e^{-\frac{1}{2\pi^2 K}} > 0$$

f) In the 3D XY model, the spin-waves reduce correlations somewhat (decaying $G(r)$ with increasing r), but in 3D, spin-waves do not destroy long-range order

Additional comment:

- In 2D, spin-waves destroy long-range order, producing power-law decay (quasi-long range order). They are not capable of producing short-range order (exponential decay).
- In 1D, spinwaves destroy long-range and quasi-long range order, producing short-range order.

$\Rightarrow$ Fluctuations become increasingly important as dimensionality is lowered.

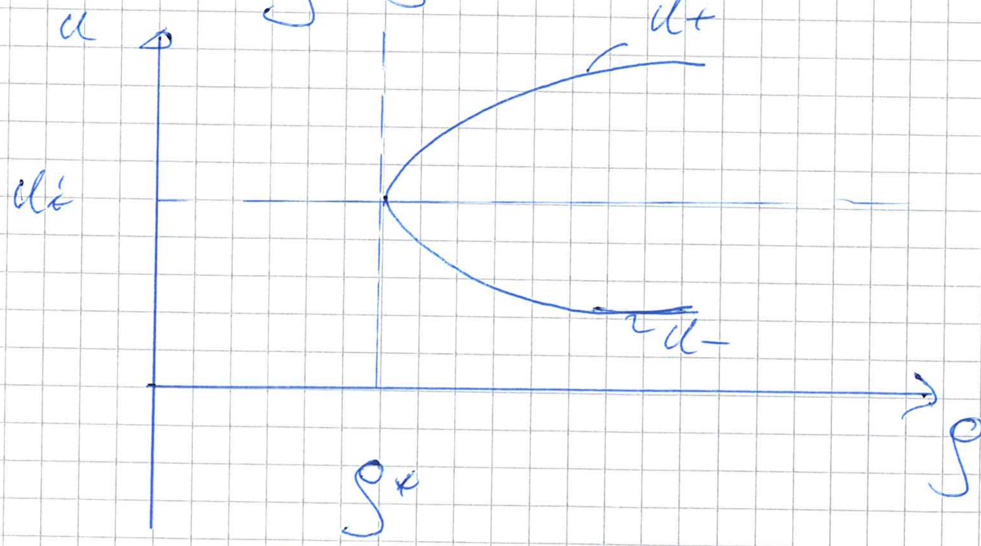
2 a) We are given a β-function for u, which tells us how u scales with length. There is no equation for g, since this is a marginal parameter.

FP: $\beta(u) = 0$

$$g - g_* = (u - u_*)^2$$

Real solution only if $g > g_*$

$$\Rightarrow \quad u_\pm = u_* \pm \sqrt{g - g_*}$$



The two fixed points $u_\pm$ vanish when $g \to g_*^*$

i) $g > g^*$: Two FP's

ii) $g < g^*$: No FP's

b) Correlation lengths Integrate β-function

i) $g > g^*$

$$\beta(u) = g - g_* - (u-u_*)^2$$

$$\varepsilon^2 \equiv g - g_* > 0$$

$$\beta(u) = -(u-u_+)(u-u_-) = \frac{du}{dl}$$

$$\frac{du}{(u-u_+)(u-u_-)} = -dl$$

$$\int_{u(0)}^{u(l)} \frac{du'}{(u'-u_+)(u'-u_-)} = -l$$

$$u_+ - u_- = 2\varepsilon = 2\sqrt{g-g_*}$$

Partial fractioning $\Rightarrow$

$$\frac{1}{2\varepsilon}\left[\ln\left|\frac{u(l)-u_+}{u(0)-u_+}\right| - \ln\left|\frac{u(l)-u_-}{u(0)-u_-}\right|\right]$$

$$= \frac{1}{2\varepsilon}\left[\ln\left|\frac{u(l)-u_+}{u(l)-u_-}\right| - \ln\left|\frac{u(0)-u_+}{u(0)-u_-}\right|\right] = -l$$

$$\xi = e^{l} f((l))$$

ξ must be l-independent

$$f(u(l)) = A\, e^{\frac{1}{2\varepsilon}\ln\left|\frac{u(l)-u_+}{u(l)-u_+}\right|}$$

With this choice of $f(u(l))$, ξ is l-independent, and given by $\frac{1}{2\varepsilon}\ln\left|\frac{u(0)-u_+}{u(0)-u_-}\right|$

$$\xi = A\, e$$

$$= A\left|\frac{u(0)-u_+}{u(0)-u_-}\right|^{\frac{1}{2\varepsilon}} \quad (\rho > \rho_c)$$

$$\left(\nu = \frac{1}{2\varepsilon} = \frac{1}{2\sqrt{\rho-\rho_c}} \quad \text{Non-universal}\right)$$

FP u_+: $u(0)-u_+=0$ Not a critical point ξ non-div.

FP u_- Critical point with a divergent length scale

(ii) $\underline{g < g^c}$

$$g^c - g = \varepsilon^2 \quad ; \quad \underline{\varepsilon \text{ small}}$$

$$\varepsilon = \sqrt{g^c - g}$$

$$\frac{du}{dl} = -\left[\varepsilon^2 + (u - u_c)^2\right]$$

$$z = \frac{u - u_c}{\varepsilon}$$

$$\frac{dz}{dl} = -\varepsilon\left(1 + z^2\right)$$

$$-\frac{dz}{z^2+1} = \varepsilon\, dl$$

$$-\frac{1}{\varepsilon}\left[\tan^{-1}(\cancel{z}(l)) - \tan^{-1}(z(0))\right] = l$$

Correlation length

$$\xi = e^{l} f(z(l))$$

$$f(z(l)) = A\, e^{\frac{1}{\varepsilon}\tan^{-1}(z(l))}$$

$$\xi(u) = A\, e^{\frac{1}{\varepsilon}\tan^{-1}\left(\frac{u - u_c}{\varepsilon}\right)}$$

$u < u_c \quad \tan'(z(l)) < 0$

No diverging length scale

$u > u_c : \quad \tan'(z(l)) > 0 \to \frac{\pi}{2} ; \ \varepsilon \to 0$

$$\xi = A e^{\frac{\pi}{2\sqrt{\rho^* - \rho}}} ; \quad \rho^* > \rho$$

d) We saw in problem 2b that

$$\nu = \frac{1}{2\sqrt{\rho - \rho^*}}$$

which is a non-universal exponent depending on ρ, which can be tuned at will without changing the character of the FP, provided $\rho > \rho^*$.

The unusual feature is the non-universality, and the reason for it is the presence of a marginal operator.