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Department of Physics

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Examination, course FY8304/FY3107
Mathematical approximation methods in physics
Wednesday December 3, 2014
Time: 09:00–13:00

Grades made public: Saturday January 3, 2015

Allowed to use: Calculator, mathematical tables.

All subproblems are given the same weight in the grading.

Problem 1:

Classification of a linear differential equation

Given the differential equation

$$y''(x) + \left(1 + \frac{x}{x^2+1}\right) y'(x) + \frac{1}{4}\left(1 + \frac{2x-1}{x^2+1}\right) y(x) = 0\ .$$

a) Find the singular points of the equation, and classify them.

b) Show that the leading asymptotic behaviour of $y(x)$ when $x \to \infty$ is:

$$y(x) \sim C_\pm\, x^{\pm\frac{1}{2}}\, \mathrm{e}^{-\frac{x}{2}}\ ,$$

where $C_\pm$ are constants.

c) Find the leading asymptotic behaviour of $y(x)$ at the other singular points (two answers for every singular point).

d) Can you find functions that have the correct asymptotic behaviour at all the singular points of the equation, and are analytical everywhere else?

Are they solutions of the equation?

Problem 2:

A boundary layer problem

Given the boundary value problem

$$\epsilon y''(x) + \mathrm{e}^{-x} y'(x) - \mathrm{e}^{y(x)} = 0 \ , \qquad y(0) = 1 \ , \quad y(1) = -1 \ ,$$

where ϵ is a small positive parameter. We calculate here to lowest order in ϵ.

a) Where is the boundary layer?

How does the thickness of the boundary layer scale with ϵ?

Give reasons for your answers.

b) Find the outer solution (which is approximately valid outside the boundary layer).

c) Find the inner solution (approximately valid inside the boundary layer).

d) Find the uniform solution (approximately valid in the whole interval $[0, 1]$).

Sketch what it looks like.

e) The WKB method can not be used for solving this boundary layer problem. Why not?

Problem 3:

Reduction of dimension

We consider the wave equation in one time dimension and d space dimensions:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi(t, x_1, x_2, \ldots, x_d) = 0 \ ,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \ .$$

Define

$$s = t^2 - x_1^2 - x_2^2 - \cdots - x_d^2 \ .$$

Assume that ϕ is a function of s alone, $\phi(t, x_1, x_2, \ldots, x_d) = F(s)$.

a) Derive an equation for $F(s)$, and solve it.

These Lorentz invariant solutions of the wave equation have no obvious physical interest.

b) Show that the general solution of the wave equation in one space dimension is

$$\phi(t, x) = f(t - x) + g(t + x) \ ,$$

where f and g are arbitrary functions.

Is this consistent with your solution to problem 3a)?