Solutions to eksamination in
FY8304/FY3107 Mathematical approximation methods in physics
Wednesday December 3, 2014

1a) The equation is

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0$$

with

$$p(x) = 1 + \frac{x}{x^2+1}\;, \qquad q(x) = \frac{1}{4}\left(1 + \frac{2x-1}{x^2+1}\right).$$

The singular points are the zeros $x = \pm\mathrm{i}$ of the denominator x^2+1, and possibly $x = \infty$.

The rational functions $p(x)$ and $q(x)$ both have simple poles at both singular points $x = \pm\mathrm{i}$. In fact,

$$p(x) = 1 + \frac{x}{(x-\mathrm{i})(x+\mathrm{i})} = 1 + \frac{C_1}{x-\mathrm{i}} + \frac{C_2}{x+\mathrm{i}}$$

with

$$C_1 = C_2 = \frac{1}{2}\;,$$

and

$$4q(x) = 1 + \frac{2x-1}{(x-\mathrm{i})(x+\mathrm{i})} = 1 + \frac{C_3}{x-\mathrm{i}} + \frac{C_4}{x+\mathrm{i}}$$

with

$$C_3 = 1 + \frac{\mathrm{i}}{2}\;, \qquad C_4 = 1 - \frac{\mathrm{i}}{2}\;.$$

Hence both $x = \mathrm{i}$ and $x = -\mathrm{i}$ are regular singular points.

To check the nature of the possible singular point $x = \infty$ we change variable to $u = 1/x$ and use that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}\,\frac{\mathrm{d}y}{\mathrm{d}u} = -\frac{1}{x^2}\,\frac{\mathrm{d}y}{\mathrm{d}u} = -u^2\,\frac{\mathrm{d}y}{\mathrm{d}u}\;, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = u^2\,\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\,\frac{\mathrm{d}y}{\mathrm{d}u}\right) = u^4\,\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 2u^3\,\frac{\mathrm{d}y}{\mathrm{d}u}\;.$$

The equation takes the following form,

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} + \left(\frac{2}{u} - \frac{p}{u^2}\right)\frac{\mathrm{d}y}{\mathrm{d}u} + \frac{q}{u^4}\,\frac{\mathrm{d}y}{\mathrm{d}u} = 0\;.$$

We have that

$$\frac{x}{x^2+1} \sim \frac{1}{x}\;, \qquad \frac{2x-1}{x^2+1} \sim \frac{2}{x}$$

as $x \to \infty$, and hence

$$\frac{p}{u^2} \sim \frac{1}{u^2}\;, \qquad \frac{q}{u^4} \sim \frac{1}{4u^4}$$

as $x \to \infty$, $u = 1/x \to 0$.

Hence $x = \infty$ is an irregular singular point of the equation.

1b) To find the leading asymptotic behaviour of $y(x)$ as $x \to \infty$ we write $y(x) = \mathrm{e}^{S(x)}$, this gives the equation

$$S''(x) + (S'(x))^2 + p(x)\,S'(x) + q(x) = 0\ .$$

We make the ansatz $S'(x) = Ax^\alpha$ with A and α constant and get that

$$A\alpha x^{\alpha-1} + A^2x^{2\alpha} + p(x)Ax^\alpha + q(x) = 0\ .$$

In the limit $x \to \infty$ we have that $p(x) \sim 1$ and $q(x) \sim 1/4$ to leading order. We may neglect $x^{\alpha-1}$ as compared to $p(x)x^\alpha$. We have to choose $\alpha = 0$, and we get the equation

$$A^2 + A + \frac{1}{4} = \left(A + \frac{1}{2}\right)^2 = 0\ ,$$

with the unique solution $A = -1/2$.

We next write $S = S_0 + S_1$ with $S_0' = -1/2$. This gives for S_1 the equation

$$S_1''(x) + \left(-\frac{1}{2} + S_1'(x)\right)^2 + p(x)\left(-\frac{1}{2} + S_1'(x)\right) + q(x) = 0\ ,$$

which we rewrite further as

$$S_1''(x) + (S_1'(x))^2 + (p(x) - 1)S_1'(x) + \frac{1}{4} - \frac{p(x)}{2} + q(x) = 0\ ,$$

and even more explicitly,

$$S_1''(x) + (S_1'(x))^2 + \frac{x}{x^2+1}\,S_1'(x) - \frac{1}{4(x^2+1)} = 0\ .$$

In this equation we make the ansatz $S_1'(x) = Bx^\beta$ and keep only the leading orders of the fractions, this gives the equation

$$B\beta x^{\beta-1} + B^2x^{2\beta} + \frac{1}{x}\,Bx^\beta - \frac{1}{4x^2} = 0\ .$$

The obvious choice is $\beta = -1$, giving the equation

$$B^2 - \frac{1}{4} = 0\ ,$$

with the two solutions $B = \pm 1/2$. Integration of the equation

$$S' = S_0' + S_1' = -\frac{1}{2} \pm \frac{1}{2x}$$

gives that

$$S = S_0 + S_1 = -\frac{x}{2} \pm \frac{1}{2}\,\ln x + \ln C_\pm$$

and

$$y(x) = \mathrm{e}^{S(x)} = C_\pm\, x^{\pm\frac{1}{2}}\,\mathrm{e}^{-\frac{x}{2}}\ .$$

1c) At a regular singular point we know that the leading asymptotic behaviour is a power. Close to the singular point $x = \mathrm{i}$ we try the approximate solution

$$y(x) = (x - \mathrm{i})^\alpha$$

hoping to find a good value for the power α. To leading order we should have that

$$\alpha(\alpha - 1)(x - \mathrm{i})^{\alpha-2} + p(x)\alpha(x - \mathrm{i})^{\alpha-1} + q(x)(x - \mathrm{i})^\alpha = 0 \ .$$

The index equation is the condition that the coefficient in front of $(x - \mathrm{i})^{\alpha-2}$ in this equation has to vanish. With

$$p(x) \sim \frac{C_1}{x - \mathrm{i}} \ , \qquad C_1 = \frac{1}{2} \ , \qquad q(x) \sim \frac{C_3}{4(x - \mathrm{i})} \ , \qquad C_3 = 1 + \frac{\mathrm{i}}{2} \ ,$$

we get the following index equation,

$$\alpha(\alpha - 1) + C_1\alpha = 0 \ ,$$

with the two solutions $\alpha = 0$ or $\alpha = 1 - C_1 = 1/2$.

The index $\alpha = 0$ defines a solution $y_1(x)$ which is analytic at $x = \mathrm{i}$ with $y_1(\mathrm{i}) \neq 0$. The other index $\alpha = 1/2$ defines a linearly independent solution $y_2(x) = \sqrt{x - \mathrm{i}}\, f(x)$, where $f(x)$ is analytic at $x = \mathrm{i}$ and $f(\mathrm{i}) \neq 0$.

A similar analysis at $x = -\mathrm{i}$ identifies another pair of linearly independent solutions, $y_3(x)$ which is analytic and nonzero at $x = -\mathrm{i}$, and $y_4(x) = \sqrt{x + \mathrm{i}}\, g(x)$, where $g(x)$ is analytic and nonzero at $x = -\mathrm{i}$.

1d) After our analysis of the asymptotic behaviours at the singular points we are now in a position to write down the following two functions that both have the correct asymptotic behaviours at all three singular points,

$$y_5(x) = \sqrt{x - \mathrm{i}}\, \mathrm{e}^{-\frac{x}{2}} \ , \qquad y_6(x) = \sqrt{x + \mathrm{i}}\, \mathrm{e}^{-\frac{x}{2}} \ .$$

It seems at least a reasonable guess that they are two linearly independent exact solutions of the equation.

The difference $y_5(x) - y_6(x)$ has the subdominant asymptotic behaviour $x^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{x}{2}}$ at $x = \infty$, because the dominant asymptotic behaviour $x^{\frac{1}{2}}\, \mathrm{e}^{-\frac{x}{2}}$ cancels when we take the difference.

In order to verify that $y_5(x)$ is a solution we compute

$$y_5' = \left(\frac{1}{2(x - \mathrm{i})} - \frac{1}{2}\right) y_5 \ ,$$

$$y_5'' = -\frac{1}{2(x - \mathrm{i})^2}\, y_5 + \left(\frac{1}{2(x - \mathrm{i})} - \frac{1}{2}\right)^2 y_5 = \left(-\frac{1}{4(x - \mathrm{i})^2} - \frac{1}{2(x - \mathrm{i})} + \frac{1}{4}\right) y_5 \ .$$

We need to verify that

$$-\frac{1}{4(x - \mathrm{i})^2} - \frac{1}{2(x - \mathrm{i})} + \frac{1}{4} + p(x)\left(\frac{1}{2(x - \mathrm{i})} - \frac{1}{2}\right) + q(x) = 0 \ .$$

We write this equation as

$$-\frac{1}{4(x-\mathrm{i})^2}+\frac{p(x)-1}{2(x-\mathrm{i})}+\frac{1}{4}-\frac{p(x)}{2}+q(x)=0\ ,$$

or more explicitly as

$$-\frac{1}{4(x-\mathrm{i})^2}+\frac{x}{x^2+1}\,\frac{1}{2(x-\mathrm{i})}-\frac{1}{4(x^2+1)}=0\ .$$

Multiplication by $4(x^2+1)=4(x-\mathrm{i})(x+\mathrm{i})$ gives the equivalent equation

$$-\frac{x+\mathrm{i}}{x-\mathrm{i}}+\frac{2x}{x-\mathrm{i}}-1=0\ ,$$

which obviously holds.

The verification that $y_6(x)$ is a solution is the same, we just replace i by $-$i.

2a) The boundary layer is at $x=0$, because the coefficients ϵ in front of y'' and e^{-x} in front of y' have the same sign, with a large ratio e^{-x}/ϵ, implying that there exists a solution which is everywhere rapidly decreasing with increasing x. The only way this solution can avoid blowing up as $x\to 0^+$ is that it is very close to zero except in a thin boundary layer close to $x=0$.

Its thickness δ scales linearly with ϵ. To see this in a more formal way we introduce the inner variables $X=x/\delta$ and $Y(X)=y(x)$ and rewrite the equation as

$$\frac{\epsilon}{\delta^2}\,Y''(X)+\frac{\mathrm{e}^{-\delta X}}{\delta}\,Y'(X)-\mathrm{e}^{Y(X)}=0\ . \tag{1}$$

The method of dominant balance tells us to choose $\delta=\epsilon$ (or δ proportional to ϵ) so that the first two terms balance, both become large of order $1/\epsilon$, and the third term becomes much smaller, of order one.

2b) In the limit $\epsilon\to 0^+$ we get the outer equation

$$\mathrm{e}^{-x}y'-\mathrm{e}^{y}=0\ ,$$

which we rewrite as

$$\mathrm{e}^{-y}y'-\mathrm{e}^{x}=0\ ,$$

and integrate to get the solution

$$-\mathrm{e}^{-y}-\mathrm{e}^{x}=C_1\ .$$

We determine the integration constant from the boundary condition $y(1)=-1$,

$$C_1=-2\mathrm{e}\ .$$

This gives the explicit outer solution

$$y_{\mathrm{out}}(x)=-\ln(2\mathrm{e}-\mathrm{e}^{x})\ .$$

2c) The inner equation is found from equation (1) in the limit $\delta = \epsilon \to 0$,

$$Y''(X) + Y'(X) = 0 \; .$$

The solution is

$$Y(X) = C_2 + C_3 \mathrm{e}^{-X} \; .$$

The boundary condition $y(0) = 1$ gives one equation for the two integration constants,

$$C_2 + C_3 = 1 \; .$$

This gives the explicit inner solution

$$y_{\text{in}}(x) = 1 + C_3(\mathrm{e}^{-\frac{x}{\epsilon}} - 1) \; .$$

2d) The inner and outer solutions have to match in the limit $\epsilon \to 0$, $x \to 0$, $x/\epsilon \to +\infty$, where

$$y_{\text{out}}(x) \to -\ln(2\mathrm{e} - 1) \; , \qquad y_{\text{in}}(x) \to 1 - C_3 \; .$$

The condition that these limits are equal determines the third integration constant,

$$C_3 = 1 + \ln(2\mathrm{e} - 1) \; .$$

The matching solution, where the inner and outer solutions match, is

$$y_{\text{match}}(x) = -\ln(2\mathrm{e} - 1) = 1 - C_3 \; .$$

The uniform solution is

$$y_{\text{uniform}}(x) = y_{\text{in}}(x) + y_{\text{out}}(x) - y_{\text{match}}(x) = (1 + \ln(2\mathrm{e} - 1))\,\mathrm{e}^{-\frac{x}{\epsilon}} - \ln(2\mathrm{e} - \mathrm{e}^{x}) \; .$$

The dotted line in Figure 1 shows our approximate solution for $\epsilon = 0.1$, compared to the exact solution, which is the full drawn line. The exact solution was computed numerically with the initial values $y(1) = -1$, $y'(1) = 0.703$.

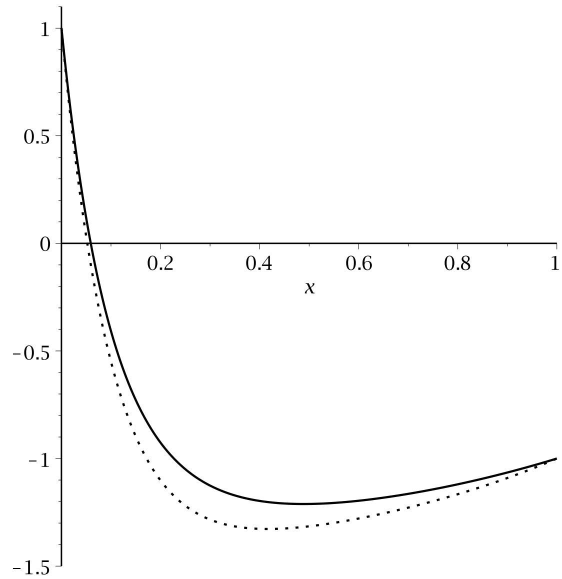
Figure 2 shows the same comparison for $\epsilon = 0.01$. The exact solution in this case was computed numerically with initial values $y(0) = 1$, $y'(0) = -246.45$.

The good way to compute the numerical solution is to integrate in the direction in which the rapidly varying solution is decreasing. If we try to integrate in the wrong direction, it can be done in Maple with 40 digits precision and with the initial values

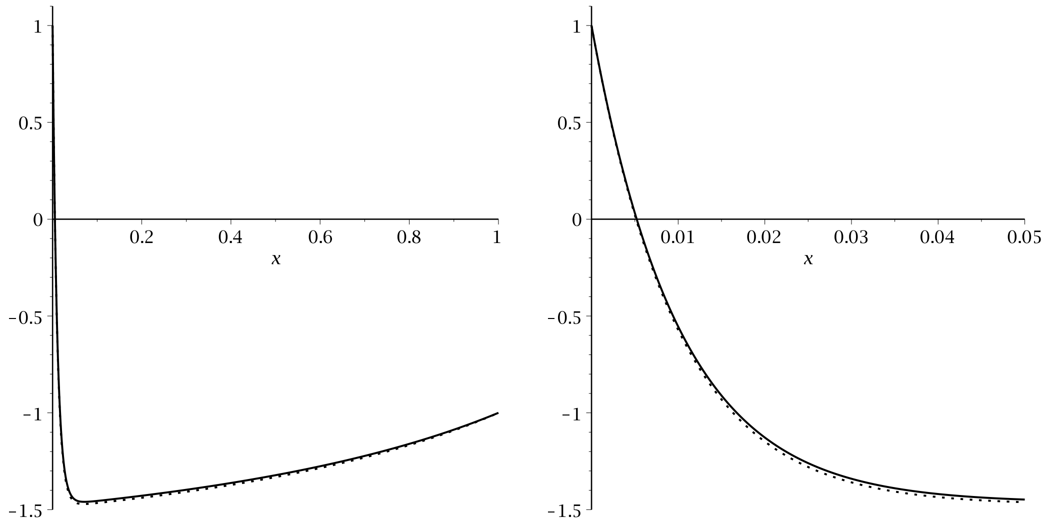
$$y(1) = -1 \; , \qquad y'(1) = 0.951\,787\,735\,074\,023\,118\,640\,723\,097\,3 \; .$$

It is necessary then to specify $y'(1)$ with a precision of 31 decimals (!) which can be found by trial and error.

2e) The WKB method can not be used here because it applies only to equations that are linear in y.



Figur 1: Exact solution (full drawn line) and approximate solution (dotted line) for $\epsilon = 0.1$.



Figur 2: Exact solution (full drawn line) and approximate solution (dotted line) for $\epsilon = 0.01$.

3a) We get that

$$\frac{\partial}{\partial t}\, F(s) = 2t\, F'(s)\ , \qquad \frac{\partial^2}{\partial t^2}\, F(s) = 4t^2 F''(s) + 2\, F'(s)\ ,$$
$$\frac{\partial}{\partial x_k}\, F(s) = -2x_k\, F'(s)\ , \qquad \frac{\partial^2}{\partial x_k^2}\, F(s) = 4x_k^2 F''(s) - 2\, F'(s)\ , \qquad k = 1, 2, \ldots, d\ .$$

Hence,

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) F(s) = 4sF''(s) + 2(d+1)F'(s) \ ,$$

and the wave equation takes the form

$$\frac{F''(s)}{F'(s)} + \frac{d+1}{2s} = 0 \ .$$

Integration gives that

$$\ln F'(s) = -\frac{d+1}{2} \ \ln s + C_1 \ , \qquad F'(s) = C_2 \, s^{-\frac{d+1}{2}} \ .$$

If $d \neq 1$ then

$$F(s) = C_3 \, s^{-\frac{d-1}{2}} + C_4 \ .$$

If $d = 1$ then

$$F(s) = C_2 \ \ln s + C_5 \ ,$$

with constants C_1, C_2, C_3, C_4, C_5.

3b) In the case $d = 1$ the wave equation is

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) \phi(t,x) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) \phi(t,x) = 0 \ .$$

If we define

$$f_1(t,x) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) \phi(t,x) \ ,$$

then the equation is

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) f_1(t,x) = 0 \ ,$$

which has the general solution

$$f_1(t,x) = f_2(t-x) \ .$$

Next, $\phi(t,x)$ is a solution of the inhomogeneous equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) \phi(t,x) = f_2(t-x) \ .$$

The general solution of the corresponding homogeneous equation is

$$\phi(t,x) = g(t+x) \ .$$

A special solution of the inhomogeneous equation is $\phi(t,x) = f(t-x)$, where

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) f(t-x) = 2f'(t-x) = f_2(t-x) \ .$$

Hence the general solution of the inhomogeneous equation is

$$\phi(t,x) = f(t-x) + g(t+x) \ .$$

Under point 3a) we found the special solution

$$F(s) = C_2 \ \ln s + C_5 = C_2 \ \ln(t^2 - x^2) + C_5 = C_2 \left(\ln(t-x) + \ln(t+x)\right) + C_5 \ ,$$

which is of the general form $f(t-x) + g(t+x)$.