# FY3107/8304 Mathematical approximation methods in physics
# Solution to exam, December 2020

A remark on the marking: The performance on subproblems 1c, 3a and 3b was considerably weaker than I had expected. In the calculation of the candidate's total percentage, I judged that it was reasonable to reduce the significance of these three subproblems (but in such a way that those few students who had done well on them were not disadvantaged by this change).

As all materials were allowed in this exam, it was natural to use various results derived in the lecture notes and/or in the textbook (Bender & Orszag, hereafter referred to as BO).

**Problem 1**

(a) We see by inspection that $x = 0$ is a singular point, and that there are no other singular points for finite $|x|$. While $x \cdot \frac{1}{4x} = \frac{1}{4}$ is analytic at $x = 0$, $x^2 \cdot \left(-\frac{1}{x^3}\right) = -\frac{1}{x}$ is not. Thus $x = 0$ is an irregular singular point.

To investigate the point $x = \infty$ (complex infinity) we define $t = 1/x$ and $Y(t) = y(x)$ to classify the point $t = 0$. This gives

$$\frac{d}{dx} = -t^2 \frac{d}{dt}, \qquad \frac{d^2}{dx^2} = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt}, \tag{1}$$

which leads to the ODE

$$\frac{d^2Y}{dt^2} + \frac{7}{4t}\frac{dY}{dt} - \frac{1}{t}Y = 0. \tag{2}$$

We see that $t = 0$ is a singular point. Since $t \cdot \frac{7}{4t} = \frac{7}{4}$ and $t^2 \cdot \left(-\frac{1}{t}\right) = -t$ are both analytic at $t = 0$, $t = 0$ is a regular singular point.

In summary, the ODE has an irregular singular point at $x = 0$ and a regular singular point at $x = \infty$.

(b) We expect the general solution to have an essential singularity at $x = 0$ since this is an irregular singular point for the ODE. Therefore we try a solution on the form $y(x) = \exp(S(x))$ (**exponential substitution**). This gives

$$y' = S'e^S, \qquad y'' = [S'' + (S')^2]e^S, \tag{3}$$

which after inserting into the ODE for $y$ and cancelling the common factor $\exp(S)$ gives

$$S'' + (S')^2 + \frac{1}{4x}S' = \frac{1}{x^3}. \tag{4}$$

This is a nonlinear ODE that we are unable to solve exactly. Instead we try to do an asymptotic analysis using the **method of dominant balance**. Based on previous experience, we first try the assumptions that $S'', (4x)^{-1}S' \ll (S')^2, 1/x^3$ $(x \to 0^+)$. This gives

$$(S')^2 \sim \frac{1}{x^3} \quad \Rightarrow \quad S' \sim \pm\frac{1}{x^{3/2}} \qquad (x \to 0^+) \tag{5}$$

Thus $S'/(4x) \sim \pm(1/4)x^{-5/2}$ and $S'' \sim \mp(3/2)x^{-5/2}$, both $\ll 1/x^3$ $(x \to 0^+)$, so the assumptions were consistent. Integrating gives

$$S(x) \sim \mp\frac{2}{\sqrt{x}} \qquad (x \to 0^+) \tag{6}$$

Next we write $S(x) = \mp 2x^{-1/2} + C(x)$. Thus $S' = \pm x^{-3/2} + C'$ and $S'' = \mp(3/2)x^{-5/2} + C''$. Inserting into (4) gives

$$\mp(3/2)x^{-5/2} + C'' + (\pm x^{-3/2} + C')^2 + \frac{1}{4x}(\pm x^{-3/2} + C') = \frac{1}{x^3}. \tag{7}$$

Expanding out, cancelling and combining terms gives the five-term nonlinear ODE

$$\mp\frac{5}{4}x^{-5/2} + C'' \pm 2C'x^{-3/2} + (C')^2 + \frac{1}{4}x^{-1}C' = 0. \tag{8}$$

Here $\frac{1}{4}x^{-1}C' \ll 2C'x^{-3/2}$ $(x \to 0^+)$ (this holds independently of $C'$). Furthermore, differentiating the relation $C(x) \ll x^{-1/2}$ (which follows from (6)) gives $C' \ll x^{-3/2}$, so $(C')^2 \ll 2C'x^{-3/2}$. Differentiating again gives $C'' \ll x^{-5/2}$. Thus we arrive at the dominant balance

$$2C'x^{-3/2} \sim \frac{5}{4}x^{-5/2} \quad \Rightarrow \quad C' \sim \frac{5}{8}x^{-1} \qquad (x \to 0^+) \tag{9}$$

It can be checked from this result that also the last two terms that were neglected were indeed negligible. Integrating (9) gives

$$C \sim \frac{5}{8}\ln x \qquad (x \to 0^+) \tag{10}$$

Continuing the analysis a little further, it can be shown that the next term in the asymptotic expansion of $S(x)$ is a constant, i.e.

$$S(x) \sim \mp\frac{2}{\sqrt{x}} + \frac{5}{8}\ln x + k_\mp \qquad (x \to 0^+) \tag{11}$$

and that the difference between the lhs and rhs in this asymptotic relation is $\ll 1$ $(x \to 0^+)$ (I omit the details here). It is therefore permissible to exponentiate the asymptotic relation (11). This gives the possible leading behaviours as

$$y_\pm(x) \sim \exp\left(\pm\frac{2}{\sqrt{x}} + \frac{5}{8}\ln x + k_\pm\right) = c_\pm x^{5/8}\exp\left(\pm\frac{2}{\sqrt{x}}\right) \qquad (x \to 0^+), \tag{12}$$

where the two constants $c_\pm = \exp(k_\pm)$ are arbitrary.

(c) The solution that goes to 0 as $x \to 0^+$ is $y_-(x)$. In the following I consider this solution only and therefore drop the subscript to lighten the notation. Setting the constant $c = 1$ (as it is of no importance in the following analysis), the solution can be written $y(x) = L(x)w(x)$, where $L(x) = x^\beta \exp(-2/\sqrt{x})$ is the leading behaviour, with $\beta = 5/8$, and $w(x)$ is the unknown function whose asymptotic expansion we seek. Thus

$$y' = Lw' + L'w, \tag{13}$$

$$y'' = Lw'' + 2L'w' + L''w, \tag{14}$$

where

$$L' = L[\beta x^{-1} + x^{-3/2}], \tag{15}$$

$$L'' = L\left[\beta(\beta-1)x^{-2} + \left(2\beta - \frac{3}{2}\right)x^{-5/2} + x^{-3}\right]. \tag{16}$$

Inserting into the ODE and cancelling $L$ gives

$$\begin{aligned} &w'' + 2(\beta x^{-1} + x^{-3/2})w' + \left[\beta(\beta-1)x^{-2} + \left(2\beta - \frac{3}{2}\right)x^{-5/2} + x^{-3}\right]w \\ &+ \frac{1}{4}x^{-1}w' + \frac{1}{4}x^{-1}(\beta x^{-1} + x^{-3/2})w - x^{-3}w = 0. \end{aligned} \tag{17}$$

Collecting and simplifying coefficients of $w'$ and $w$ gives

$$w'' + \left[\left(2\beta + \frac{1}{4}\right)x^{-1} + 2x^{-3/2}\right]w' + \left[\beta\left(\beta - \frac{3}{4}\right)x^{-2} + \left(2\beta - \frac{5}{4}\right)x^{-5/2}\right]w = 0. \tag{18}$$

Inserting $\beta = 5/8$ gives the final form of the ODE for $w(x)$:

$$w'' + \left(\frac{3}{2}x^{-1} + 2x^{-3/2}\right)w' - \frac{5}{64}x^{-2}w = 0. \tag{19}$$

We now write $w(x) \sim \sum_{n=0}^{\infty} a_n x^{\alpha n}$ $(x \to 0^+)$ where $a_0 = 1$. Inserting into the ODE gives

$$\sum_n \alpha n(\alpha n - 1)a_n x^{\alpha n - 2} + \frac{3}{2}\sum_n \alpha n a_n x^{\alpha n - 2} + 2\sum_n \alpha n a_n x^{\alpha n - 5/2} - \frac{5}{64}\sum_n a_n x^{\alpha n - 2} = 0. \tag{20}$$

We anticipate that $\alpha > 0$ since the expansion point is $x = 0$. The $n = 0$ term in the last sum is $-(5/64)x^{-2}$. As the $x^{-2}$ terms vanish in the first and second sum, this leading contribution must be cancelled by the third sum. The leading term in this sum comes from $n = 1$, with exponent $\alpha \cdot 1 - 5/2$. Thus the exponents must be equal, i.e. $\alpha - 5/2 = -2$, which gives

$$\alpha = 1/2. \tag{21}$$

All sums except the third involve powers $x^{n/2-2}$. To get the third sum on this form as well we write its exponent as $n/2 - 5/2 \equiv m/2 - 2$, which gives $m = n - 1$. Thus the third term can be rewritten as

$$\sum_{n=0}^{\infty} n a_n x^{n/2-5/2} = \sum_{n=1}^{\infty} n a_n x^{n/2-5/2} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^{m/2-2} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n/2-2}, \tag{22}$$

where we renamed $m \to n$ in the last transition. Now the coefficients of $x^{n/2-2}$ can be compared, giving

$$\frac{n}{2}\left(\frac{n}{2} - 1\right) a_n + \frac{3}{4} n a_n + (n+1) a_{n+1} - \frac{5}{64} a_n = 0. \tag{23}$$

Solving for $a_{n+1}$ gives

$$a_{n+1} = \frac{\frac{5}{64} - \frac{n}{2}\left(\frac{n}{2} - 1\right) - \frac{3}{4}n}{n+1} a_n \qquad (n = 0, 1, 2, \ldots) \tag{24}$$

Starting from $a_0 = 1$, all coefficients $a_n$ can be obtained from this recursion relation.[1]

(d) Since $x = \infty$ is a regular singular point, we use the **Frobenius method** to analyze the solutions of the corresponding ODE (2) about $t = 1/x = 0$. It will be useful to consider the ODE on the form

$$\frac{d^2Y}{dt^2} + \frac{p(t)}{t}\frac{dY}{dt} + \frac{q(t)}{t^2} Y = 0, \tag{27}$$

where $p(t)$ and $q(t)$ are analytic functions at $t = 0$. Seeking a solution on Frobenius series form, i.e.

$$Y(t) = t^{\nu} \sum_{n=0}^{\infty} b_n t^n \tag{28}$$

leads to the so-called indicial equation for $\nu$:

$$\nu^2 + (p_0 - 1)\nu + q_0 = 0, \tag{29}$$

where the zeroth-order coefficients $p_0$ and $q_0$ in the Taylor series for $p(t)$ and $q(t)$ can be read off by comparing (2) and (27). This gives

$$p_0 = 7/4, \qquad q_0 = 0, \tag{30}$$

so the indicial equation becomes

$$\nu(\nu + 3/4) = 0, \tag{31}$$

which has solutions

$$\nu_1 = 0, \qquad \nu_2 = -3/4. \tag{32}$$

Since $\nu_1 - \nu_2$ is not an integer, both the linearly independent solutions will be on Frobenius series form. The leading behaviour is given by the first term $t^{\nu}$ in each solution, so the possible leading behaviours are $d_1 t^0 = d_1$ and $d_2 t^{-3/4}$ $(t \to 0)$, i.e. $d_1$ and $d_2 x^{3/4}$ $(x \to \infty)$, where $d_1$ and $d_2$ are arbitrary constants.

[1] With a bit more work, a more explicit result could be obtained by first factorizing the numerator, giving

$$a_{n+1} = -\frac{1}{4}\frac{(n - 1/4)(n + 5/4)}{n+1} a_n, \tag{25}$$

and then using $\Gamma(1 + w) = w\Gamma(w)$ to derive

$$a_n = (-1)^n \frac{\Gamma(n - 1/4)\Gamma(n + 5/4)}{4^n n! \Gamma(-1/4)\Gamma(5/4)}. \tag{26}$$

**Problem 2**

(a) We first rewrite the integral as

$$\int_0^2 dt\ \cos[x(t^2 - t)] = \mathrm{Re}\left\{\int_0^2 dt\, e^{ix(t^2-2t)}\right\}. \tag{33}$$

The integral inside curly brackets is a Fourier integral $\int_a^b dt\, f(t)\, \exp(ix\psi(t)) \equiv I(x)$ with $f(t) = 1$ and $\psi(t) = t^2 - 2t$. From $\psi'(t) = 2(t-1)$ it follows that $\psi'(t) = 0$ at the point $t = 1$, which is part of the integration interval and is therefore a stationary point of the integral. Therefore we can find the leading behaviour as $x \to +\infty$ from the **method of stationary phase**. This gives that the Fourier integral $I(x)$ with a single stationary point at the left endpoint $t = a$ has leading behaviour[2]

$$I(x) \sim f(a)e^{ix\psi(a)\pm i\pi/(2p)}\left(\frac{p!}{x|\psi^{(p)}(a)|}\right)^{1/p}\frac{\Gamma(1/p)}{p} \qquad (x \to +\infty) \tag{34}$$

Here $\psi^{(p)}(a)$ is the lowest-order nonzero derivative at the stationary point, with $\pm$ being its sign. In our integral the stationary point is not an endpoint but an interior point. Since our integrand is symmetric around the stationary point $t = 1$, this just doubles the result. We have $\psi''(t) = 2$, so $p = 2$ and the sign is $+$. Also using $\psi(1) = -1$, $f(1) = 1$, and $\Gamma(1/2) = \sqrt{\pi}$, we find that the leading behaviour is

$$2\,\mathrm{Re}\left\{e^{-ix+i\pi/4}\left(\frac{2!}{x\cdot 2}\right)^{1/2}\frac{\sqrt{\pi}}{2}\right\} = \sqrt{\frac{\pi}{x}}\cos\left(x - \frac{\pi}{4}\right) \qquad (x \to +\infty) \tag{35}$$

(b) Although the integral formally can be written as a Laplace integral $\int_a^b f(t)e^{x\phi(t)}$, Laplace's method cannot be used directly since $f(t) = e^{-1/t^2}$ vanishes exponentially fast at $t = 0$ where the function $\phi(t) = -t$ has its maximum. Instead we must treat this as an integral with a **movable** (i.e. $x$-dependent) **maximum**. Thus we consider the full integrand $\exp(\Phi(t))$ with $\Phi(t) \equiv -xt - 1/t^2$. Since $\Phi(t) \to -\infty$ both for $t \to 0$ and for $t \to \infty$, it must have a maximum somewhere inbetween, which can be found from

$$0 = \Phi'(t) = -x + 2/t^3 \quad \Rightarrow \quad t = \left(\frac{2}{x}\right)^{1/3} \equiv t_0. \tag{36}$$

We can then find the leading behaviour by expanding $\Phi(t)$ to second order around its maximum $t_0$, i.e. writing $\Phi(t) \approx \Phi(t_0) + \frac{1}{2}\Phi''(t_0)(t-t_0)^2$, and extending the integration limits to $\pm\infty$, which gives a Gaussian integral. We find

$$\Phi(t_0) = -t_0(x + t_0^{-3}) = -\left(\frac{2}{x}\right)^{1/3} x(1 + 1/2) = -3\left(\frac{x}{2}\right)^{2/3}, \tag{37}$$

$$\Phi''(t) = -6t^{-4} \quad \Rightarrow \quad \Phi''(t_0) = -6\left(\frac{x}{2}\right)^{4/3}. \tag{38}$$

Thus the leading behaviour of our integral becomes

$$\begin{aligned}\exp(\Phi(t_0))\int_{-\infty}^{\infty} d(t-t_0)\, \exp\left(-\frac{1}{2}|\Phi''(t_0)|(t-t_0)^2\right) &= \exp(\Phi(t_0))\sqrt{\frac{2\pi}{|\Phi''(t_0)|}} \\ &= \sqrt{\frac{\pi}{3}}\left(\frac{x}{2}\right)^{-2/3}\exp\left(-3\left(\frac{x}{2}\right)^{2/3}\right).\end{aligned} \tag{39}$$

(Here we used the Gaussian integral formula $\int_{-\infty}^{\infty} du\, \exp(-\frac{1}{2}|a|u^2) = \sqrt{2\pi/|a|}$.) The same result would have been obtained by first changing integration variable to $s$ via $t = st_0$, which transforms the movable maximum in $t$ to a fixed maximum at $s = 1$, and then proceeding with Laplace's method as usual.

[2] BO, Eq. (6.5.12).

(c) When applicable, using **Watson's lemma** is probably the simplest way to find the full asymptotic expansion. Motivated by this, we change integration variable to $s = \sinh(2t)$. This gives $ds = 2\cosh(2t)dt = 2\sqrt{1+s^2}dt$, where we used $\cosh^2 u - \sinh^2 u = 1$ and $\cosh u > 0$. This transforms the integral to

$$\frac{1}{2}\int_0^\infty ds\, f(s)e^{-xs} \quad \text{with} \quad f(s) = \frac{1}{\sqrt{1+s^2}}. \tag{40}$$

The function $f(s)$ has a Taylor series expansion around $s = 0$ given by

$$f(s) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{\Gamma(k+1)\Gamma(1/2)}(-1)^k s^{2k}. \tag{41}$$

We give two derivations of (41).

1. It is convenient to define $g(v) \equiv (1+v)^{-1/2}$ so that $f(s) = g(v)$ with $v = s^2$. One finds

$$g'(v) = (-1/2)(1+v)^{-3/2} \quad \Rightarrow \quad g'(0) = -\frac{1}{2^1}, \tag{42}$$

$$g''(v) = (-1/2)(-3/2)(1+v)^{-5/2} \quad \Rightarrow \quad g''(0) = (-1)^2\frac{1\cdot 3}{2^2}, \tag{43}$$

$$g'''(v) = (-1/2)(-3/2)(-5/2)(1+v)^{-7/2} \quad \Rightarrow \quad g'''(0) = (-1)^3\frac{1\cdot 3\cdot 5}{2^3}, \tag{44}$$

   etc. The pattern is clear, giving

$$g^{(k)}(0) = (-1)^k\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k} = (-1)^k\frac{\Gamma(k+1/2)}{\Gamma(1/2)}, \tag{45}$$

   where the last transition can be shown[3] by repeated use of $\Gamma(w+1) = w\Gamma(w)$. Eq. (41) now follows from Taylor's theorem: $g(v) = \sum_{k=0}^{\infty}\frac{1}{k!}g^{(k)}(0)v^k$.

2. Use the binomial theorem $(1+v)^r = \sum_{k=0}^{\infty}\binom{r}{k}v^k$ with $r = -1/2$ and $v = s^2$. Since in our case $r$ is a fraction, the factorials in the binomial coefficient $\binom{r}{k} = \frac{r!}{k!(r-k)!}$ should be understood in terms of the $\Gamma$ function via $n! = \Gamma(n+1)$. This gives

$$f(s) = \sum_{k=0}^{\infty}\frac{\Gamma(1/2)}{\Gamma(k+1)\Gamma(-k+1/2)}s^{2k}. \tag{46}$$

   Here the sign of $\Gamma(-k+1/2)$ will oscillate with $k$. To make this oscillation explicit we use the relation $\Gamma(a)\Gamma(1-a) = \pi/\sin(\pi a)$ given in the problem text, with $a = -k+1/2$, to write

$$\Gamma(-k+1/2) = \frac{\pi}{\sin[\pi(1/2-k)]}\frac{1}{\Gamma(k+1/2)} = \frac{(\Gamma(1/2))^2}{(-1)^k\Gamma(k+1/2)}, \tag{47}$$

   where we also used $\Gamma(1/2) = \sqrt{\pi}$ and $\sin(\pi/2 - \pi k) = \cos(\pi k) = (-1)^k$. Inserting (47) into (46) gives (41).

The asymptotic expansion can now be obtained from Watson's lemma, which involves making two approximations that can be shown to only introduce exponentially small errors: (i) reduce the upper integration limit to a value $\epsilon > 0$ small enough that the Taylor series for $f(s)$ is valid, (ii) extend the integration limit from $\epsilon$ up to $\infty$, to enable the evaluation of the remaining integral:

$$\int_0^\infty ds\, s^{2k}e^{-xs} \stackrel{t=xs}{=} x^{-(2k+1)}\int_0^\infty dt\, t^{2k}e^{-t} = x^{-(2k+1)}\Gamma(2k+1). \tag{48}$$

[3] This result was derived on p. 15 in the lecture notes on the method of steepest descent.

(d) Denote the terms in the asymptotic expansion by $f_k$. The rule for the optimal asymptotic approximation is to truncate the series such that the first omitted term has the smallest magnitude of all terms. Based on the expression for $f_k$ and the typical behaviour of the terms of an asymptotic series, we expect that, at least for sufficiently large $x$, $|f_k|$ will first decrease with $k$ until reaching a minimum at $k = k_0$, and then increase without bound for larger $k$. Correspondingly, we expect $|f_{k+1}/f_k|$ to increase with $k$ from a starting value below 1. Thus the minimum $k_0$ of $f_k$ can be estimated from where $|f_{k+1}/f_k|$ crosses 1.[4] Thus we consider

$$\begin{aligned} \left|\frac{f_{k+1}}{f_k}\right| &= \frac{\frac{\Gamma(2(k+1)+1)\Gamma(k+1+1/2)}{\Gamma(k+1+1)}}{\frac{\Gamma(2k+1)\Gamma(k+1/2)}{\Gamma(k+1)}} \cdot \frac{x^{-2(k+1)}}{x^{-2k}} = \frac{\Gamma(2k+3)}{\Gamma(2k+1)} \cdot \frac{\Gamma(k+3/2)}{\Gamma(k+1/2)} \cdot \frac{\Gamma(k+1)}{\Gamma(k+2)} \cdot x^{-2} \\ &= (2k+2)(2k+1) \cdot (k+1/2) \cdot \frac{1}{k+1} \cdot x^{-2}, \end{aligned} \tag{49}$$

where we used $\Gamma(w+1) = w\Gamma(w)$ in the last transition to simplify the ratios of $\Gamma$ functions. Next, as $k_0$ will be large for $x$ large, we may neglect the small additive constants in each of the $k$-dependent factors. This gives

$$\left|\frac{f_{k+1}}{f_k}\right| \approx \frac{(2k)(2k)k}{k} x^{-2} = \frac{4k^2}{x^2}. \tag{50}$$

Setting this equal to 1 gives

$$k_0 \approx \frac{x}{2}. \tag{51}$$

**Problem 3**

(a) For $x \leq 0$ the potential $V(x) = \infty$, so the wavefunction $y(x)$ must vanish. Thus, in particular, $y(0) = 0$. Therefore, since the wavefunction $y(x)$ should be continuous, the approximation for $y(x)$ for $x > 0$ must approach 0 smoothly as $x \to 0$. The calculation for the problem with one simple turning point gives the following WKB approximation for $y(x)$ to the left of the turning point $x_T$:[5]

$$y_{\mathrm{WKB}}(x) \sim 2C[-Q(x)]^{-1/4} \sin\left[\frac{1}{\epsilon}\int_x^{x_T} dt\,\sqrt{-Q(t)} + \frac{\pi}{4}\right] \tag{52}$$

where $C$ is a normalization constant. Since for our $V(x)$, the function $Q(x) = V(x) - E$ does not go to zero as $x$ approaches 0 from the right, the approximation (52) is valid arbitrarily close to $x = 0$. It follows that we should impose the condition $y_{\mathrm{WKB}}(0) = 0$. This can only be satisfied if the sine function in (52) vanishes for $x = 0$, i.e. if

$$\frac{1}{\epsilon}\int_0^{x_T} dt\,\sqrt{-Q(t)} + \frac{\pi}{4} = m\pi \quad \Rightarrow \quad \frac{1}{\epsilon}\int_0^{x_T} dt\,\sqrt{-Q(t)} = \left(m - \frac{1}{4}\right)\pi, \tag{53}$$

where $m$ is an integer. Furthermore, as the integral is positive, $m - 1/4$ must be positive, so the possible values of $m$ are restricted to $m = 1, 2, 3, \ldots$. Since the problem text asks for a labeling of the eigenvalues starting from quantum number 0, we define $n = m - 1$, which gives the eigenvalue condition

$$\frac{1}{\epsilon}\int_0^{x_T} dt\,\sqrt{-Q(t)} = \left(n + \frac{3}{4}\right)\pi \qquad (n = 0, 1, 2, \ldots) \tag{54}$$

(b) The Schrödinger equation for a one-dimensional harmonic oscillator with mass $m$ and angular frequency $\omega$ is

$$-\frac{\hbar^2}{2m}\frac{d^2}{d\tilde{x}^2}\psi(\tilde{x}) + \frac{1}{2}m\omega^2\tilde{x}^2\psi(\tilde{x}) = \tilde{E}\psi(\tilde{x}) \tag{55}$$

[4] This is technically easier than finding the minimum of $f_k$ from $df_k/dk = 0$, since that would require invoking the asymptotic (Stirling) approximation for the $\Gamma$ functions with a large argument.

[5] This result was derived in the lecture notes, where the turning point was called $x_2$; see p. 8 in the file "WKB theory 3". The same result can also be found in Eq. (10.4.13c) in BO, (there the turning point was taken to be at $x_T = 0$).

where $\tilde{x}$ is the physical position (of dimension length) and $\tilde{E}$ is the physical energy (of dimension energy). As shown in detail in the lecture notes,[6] this Schrödinger equation can be rewritten as

$$y''(x) = \left(\frac{1}{4}x^2 - E\right) y(x) \tag{56}$$

where $x$ and $E$ are dimensionless variables given by $E = \tilde{E}/(\hbar\omega)$ and $x = \tilde{x}/\ell$, where $\ell$ is the length scale $\ell = \sqrt{\hbar/(2m\omega)}$. This derivation is clearly also valid for $x > 0$ for the potential $V(x)$ considered here. It follows that $\epsilon = 1$, $f(x) = x^2/4$, and $x_T = 2\sqrt{E}$. Inserting this into the eigenvalue condition (54) gives

$$\int_0^{2\sqrt{E}} dt\, \sqrt{E - x^2/4} = (n + 3/4)\pi. \tag{57}$$

The integral equals $E\pi/2$.[7] Thus

$$E_n = 2n + \frac{3}{2} \quad \Rightarrow \quad \tilde{E}_n = \hbar\omega\left(2n + \frac{3}{2}\right) \qquad (n = 0, 1, 2, \ldots) \tag{58}$$

Comments:

- The analysis here is based on the physical-optics approximation. This approximation is valid for (i) $\epsilon \to 0$ with $E$ fixed, or for (ii) $E \to \infty$ (i.e. $n \to \infty$) with $\epsilon$ fixed; see BO p. 521. Since $\epsilon = 1$ in our application here, the condition (i) is obviously not satisfied, so we would in general expect the analysis to be valid only as $n \to \infty$. However, the prediction (58) for the eigenvalues $E_n$ turns out to be correct for all $n = 0, 1, 2, \ldots$.

- The result (58) can be understood from the perspective of the standard one-dimensional harmonic oscillator problem. To avoid confusion with the quantum number $n$, let the quantum number for the standard harmonic oscillator be $N = 0, 1, 2, \ldots$, so that its eigenvalues are $E_N = \hbar\omega(N + 1/2)$ and the associated eigenfunctions are $y_N(x)$. Recall that the eigenfunctions for odd $N$, i.e. $N = 1, 3, 5, \ldots$, are odd in $x$ and thus satisfy $y_N(0) = 0$. Thus for $x > 0$ these will be exact eigenfunctions also for the potential $V(x)$ studied here. So $E_N = N + 1/2$ with $N$ restricted to $N = 1, 3, 5, \ldots$ are eigenvalues of $V(x)$. Relabeling by defining $n$ via $N = 2n + 1$, the new label $n$ runs over $n = 0, 1, 2, \ldots$ with associated energy $(2n+1)+1/2 = 2n+3/2$, consistent with (58).

**Problem 4**

(a) The solution of this problem involves a **multiple-scale analysis**. (We take $y$ to be real, as in the other examples of multiple-scale analysis we have discussed.) From $y(t) \sim Y_0(t, \tau) + \epsilon Y_1(t, \tau) + \ldots$ with $\tau = \epsilon t$ we get

$$y'(t) = \frac{dy}{dt} \sim \frac{\partial Y_0}{\partial t} + \epsilon\left(\frac{\partial Y_0}{\partial \tau} + \frac{\partial Y_1}{\partial t}\right), \tag{59}$$

$$y''(t) = \frac{d^2y}{dt^2} \sim \frac{\partial^2 Y_0}{\partial t^2} + \epsilon\left(2\frac{\partial^2 Y_0}{\partial \tau \partial t} + \frac{\partial^2 Y_1}{\partial t^2}\right). \tag{60}$$

Inserting into the ODE and comparing coefficients of equal powers of $\epsilon$ gives

$$O(\epsilon^0): \qquad \frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{61}$$

$$O(\epsilon^1): \qquad \frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial \tau \partial t} - Y_0^2 \frac{\partial Y_0}{\partial t}. \tag{62}$$

The general real solution to (61) can be written (here $^*$ denotes the complex conjugate)

$$Y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it} \tag{63}$$

[6] See p. 12-13 in the file "WKB theory 3".
[7] See e.g. the almost identical calculation in the lecture notes; p. 14 in the file "WKB theory 3".

where the function $A(\tau)$ remains to be determined. It follows from (63) that

$$\frac{\partial Y_0}{\partial t} = i(Ae^{it} - A^* e^{-it}), \quad (64)$$

$$\frac{\partial^2 Y_0}{\partial \tau \partial t} = i\left(\frac{dA}{d\tau}e^{it} - \frac{dA^*}{d\tau}e^{-it}\right). \quad (65)$$

Inserting these expressions into the rhs of (62), the problematic terms are the secular terms, which involve the $t$-dependence $e^{\pm it}$. To prevent their appearance one sets their coefficients to 0, which gives the following ODE for $A(\tau)$:

$$\frac{dA}{d\tau} = -\frac{1}{2}A^2 A^*. \quad (66)$$

(b) To solve this ODE we write $A(\tau)$ on polar form, i.e. $A(\tau) = R(\tau)e^{i\theta(\tau)}$ with $R$ and $\theta$ real with $R \geq 0$. This gives, after cancelling the common factor $e^{i\theta(\tau)}$ and separating the real and imaginary parts, the two ODEs

$$\frac{dR}{d\tau} = -\frac{1}{2}R^3, \qquad \frac{d\theta}{d\tau} = 0, \quad (67)$$

whose solutions are

$$R(\tau) = \frac{R(0)}{\sqrt{\tau R^2(0) + 1}}, \qquad \theta(\tau) = \theta(0). \quad (68)$$

Thus

$$Y_0(t, \tau) = 2R(\tau)\cos(t + \theta(\tau)) = \frac{2R(0)}{\sqrt{\tau R^2(0) + 1}}\cos(t + \theta(0)). \quad (69)$$

The boundary condition $y(0) = 0$ together with $y \sim Y_0$ gives $Y_0(0, 0) = 0$, i.e.

$$\cos(\theta(0)) = 0. \quad (70)$$

The boundary condition $y'(0) = 1$, together with $y' \sim \partial Y_0/\partial t = -2R(\tau)\sin(t+\theta(\tau))$, gives $(\partial Y_0/\partial t)(0, 0) = 1$, i.e.

$$-2R(0)\sin(\theta(0)) = 1. \quad (71)$$

Thus

$$\theta(0) = -\pi/2 \qquad \text{and} \qquad R(0) = 1/2, \quad (72)$$

giving

$$Y_0(t, \tau) = \frac{\sin t}{\sqrt{1 + \epsilon t/4}}. \quad (73)$$

This is the solution for $y(t)$ to leading order in $\epsilon$ in the multi-scale expansion.