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Examination for FY3452 Gravitation and Cosmology
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Possible languages for your answers: Bòkmal, Castellano, English, Nynorsk.
Allowed tools: Pocket calculator, mathematical tables
Some formulas can be found at the end of p.2.

1. Hyperbolic plane H^2.
The line-element of the Hyperbolic plane H^2 is given by

$$ds^2 = y^{-2}(dx^2 + dy^2) \qquad \text{and} \quad y \geq 0\,.$$

a. Write out the geodesic equations and deduce the Christoffel symbols $\Gamma^a{}_{bc}$. (4 pts)
b. Calculate the Riemann (or curvature) tensor $R^a{}_{bcd}$ and the scalar curvature R. (4 pts)

a. Using as Lagrange function L the kinetic energy T instead of the line-element ds makes calculations a bit shorter. From $L = y^{-2}(\dot{x}^2 + \dot{y}^2)$ we find as solutions of the Lagrange equations

$$\ddot{x} - 2\frac{\dot{x}\dot{y}}{y} = 0 \qquad \text{and} \qquad \ddot{y} - \frac{\dot{y}^2}{y} + \frac{\dot{x}^2}{y} = 0\,.$$

Comparing with the given geodesic equation, we read off the non-vanishing Christoffel symbols as $-\Gamma^x{}_{xy} = -\Gamma^x{}_{yx} = \Gamma^y{}_{xx} = -\Gamma^y{}_{yy} = 1/y$. (Remember that $-2y^{-1}\dot{x}\dot{y} = \Gamma^x{}_{xy}\dot{x}\dot{y} + \Gamma^x{}_{xy}\dot{x}\dot{y}$.)

b. We calculate e.g.

$$\begin{aligned} R^y{}_{xyx} &= \partial_y\Gamma^y{}_{xx} - \partial_x\Gamma^y{}_{xy} + \Gamma^y{}_{ey}\Gamma^e{}_{xx} - \Gamma^y{}_{ex}\Gamma^e{}_{xy} \\ &= -1/y^2 + 0 + \Gamma^y{}_{yy}\Gamma^y{}_{xx} - \Gamma^y{}_{xx}\Gamma^x{}_{xy} \\ &= -1/y^2 + 0 - 1/y^2 + 1/y^2 = -1/y^2\,. \end{aligned}$$

Next we remember that the number of independent components of the Riemann tensor in $d = 2$ is one, i.e. we are already done: All other components follow by the symmetry properties.
The scalar curvature is (diagonal metric with $g^{xx} = g^{yy} = y^2$)

$$R = g^{ab}R_{ab} = g^{xx}R_{xx} + g^{yy}R_{yy} = y^2(R_{xx} + R_{yy})\,.$$

Thus we have to find only the two diagonal components of the Ricci tensor $R_{ab} = R^c{}_{acb}$. With

$$\begin{aligned} R_{xx} &= R^c{}_{xcx} = R^x{}_{xxx} + R^y{}_{xyx} = 0 + R^y{}_{xyx} = -1/y^2 \\ R_{yy} &= R^c{}_{ycy} = R^x{}_{yxy} + R^x{}_{yxy} = R^x{}_{yxy} + 0 = -R^y{}_{xxy} = R^x{}_{yxy} = -1/y^2\,, \end{aligned}$$

the scalar curvature follows as $R = -2$. Hence the hyperbolic plane H^2 is a space of constant curvature, as $\mathbb{R}^2$ and S^2.
[If you wonder that $R = -2$, not -1: in $d = 2$, the Gaussian curvature K is connected to the

"general" scalar curvature R via $K = R/2$. Thus $K = \pm 1$ means $R = \pm 2$ for spaces of constant unit curvature radius, S^2 and H^2. You may also check that the Riemann and Ricci tensor satisfy the relations for maximally symmetric spaces, $R_{ab} = K g_{ab}$ and $R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc})$.]

2. Kerr metric.

The metric outside a spherically symmetric mass distribution with mass M and angular momentum J is given by

$$ds^2 = -\left[1 - \frac{2Mr}{\rho^2}\right]dt^2 - \frac{4Mar\sin^2\vartheta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\vartheta^2 + \left[r^2 + a^2 + \frac{2Mra^2\sin^2\vartheta}{\rho^2}\right]\sin^2\vartheta d\phi^2\,,$$

with

$$a = \frac{J}{M}\,, \qquad \rho^2 = r^2 + a^2\cos^2\vartheta\,, \qquad \Delta = r^2 - 2Mr + a^2\,.$$

a. Find the outer boundary of the ergosphere, i.e. the surface enclosing the region where no stationary observers are possible in the Kerr metric. (3 pts)
b. Find the two horizons of the Kerr metric. (1.5 pt)
d. Determine the smallest possible unstable circular orbit of a massive particle for $J = 0$. (Hint: Consider the effective potential V_{eff}.) (6 pt)

a. The normalization condition $\mathbf{u} \cdot \mathbf{u} = -1$ is inconsistent with $u^a = (1, 0, 0, 0)$ and $g_{tt} > 0$. Solving

$$g_{tt} = 1 - \frac{2Mr}{\rho^2} = 0$$

we find the position of the two stationary limit surfaces at

$$r_{1/2} = M \pm \sqrt{M^2 - a^2\cos^2\vartheta}\,. \tag{1}$$

The ergosphere is the space bounded by these two surfaces; the outer boundary corresponds to the plus sign.

b. The coordinate singularity at $\Delta = r^2 - 2Mr + a^2 = 0$ or

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

corresponds to horizons, i.e. satisfy the conditions $g^{rr} = 0$ or $g_{rr} = 1/g^{rr} = \infty$. Hence, r_- and r_+ define an inner and outer horizon around a Kerr black hole.

c. non-existent.

d. The condition $J = 0$ gives the Schwarzschild metric. Spherical symmetry allows us to choose $\vartheta = \pi/2$ and $u_\vartheta = 0$. Then we replace in the normalization condition $\mathbf{u} \cdot \mathbf{u} = -1$ written out for the Schwarzschild metric,

$$-1 = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2$$

the velocities u_t and u_r by the conserved quantities

$$\begin{aligned} e &\equiv -\xi\cdot\mathbf{u} = \left(1-\frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ l &\equiv \eta\cdot\mathbf{u} = r^2\sin\vartheta^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,. \end{aligned}$$

Inserting e and l, then reordering gives

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}$$

with

$$V_{\text{eff}} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}\,.$$

Circular orbits correspond to $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ with

$$r_{1,2} = \frac{l^2}{2M}\left[1\pm\sqrt{1-12M^2/l^2}\right]\,.$$

The unstable circular orbit (i.e. at the maximum of V_{eff}) corresponds to the minus sign, and its radius becomes smaller for $l\to\infty$. Hence

$$r_{\max} = \frac{l^2}{2M}\left[1-1+6(M/l)^2+\ldots\right] = 3M$$

is the minimum possible radius.

3. Scalar fields in FLRW metric.

Consider a scalar field ϕ with potential V

$$\mathcal{L} = \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi + V(\phi)$$

in a flat FLRW metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\sin^2\vartheta\mathrm{d}\phi^2+\mathrm{d}\vartheta^2)\right]\,.$$

a. Derive the equation of motions for ϕ. (4 pts)

b. Derive the energy-momentum tensor for ϕ. (3 pts)

c. Derive the equation of state $w = P/\rho$ for ϕ assuming that the field ϕ is uniform in space, $\phi(t,\vec{x}) = \phi(t)$. (2 pts)

d. Scalar fields are often used as models for inflation. Give *one* necessary condition that ϕ can drive inflation. (1 pt)

a. Flat means $k=0$ and thus life becomes easier using Cartesian coordinates. Then $g_{ab} = \mathrm{diag}(-1,a^2,a^2,a^2)$, $g^{ab} = \mathrm{diag}(1,a^{-2},a^{-2},a^{-2})$, and $\sqrt{|g|} = a^3$. We can use either the Lagrange formalism or (faster) use directly the action principle. Varying the action

$$S_{\text{KG}} = \int_\Omega \mathrm{d}^4x\,\sqrt{|g|}\left\{\frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi + V(\phi)\right\} = \int_\Omega \mathrm{d}^4x\,a^3\left\{-\frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\nabla\phi)^2 + V(\phi)\right\}$$

w.r.t. the field ϕ gives

$$\delta S_{\mathrm{KG}} = \int_\Omega \mathrm{d}^4x\, a^3 \left\{ -\dot\phi\delta\dot\phi + \frac{1}{a^2}(\nabla\phi)\cdot\delta(\nabla\phi) + V'\delta\phi \right\}.$$

Now we use $\delta\partial_a\phi = \partial_a\delta\phi$ (this is not an assumption, as you may convince yourself by writing this out with $\delta\phi(x) = \varepsilon\phi(x)$) and integrate the first two terms by part. The boundary terms are zero, since we require $\delta\phi(\Omega) = 0$. Finally we use $a = a(t)$ and get

$$\begin{aligned}\delta S_{\mathrm{KG}} &= \int_\Omega \mathrm{d}^4x \left\{ \frac{\mathrm{d}}{\mathrm{d}t}(a^3\dot\phi) - a\nabla^2\phi + a^3V' \right\}\delta\phi \\ &= \int_\Omega \mathrm{d}^4x\, a^3 \left\{ \ddot\phi + 3H\dot\phi - \frac{1}{a^2}\nabla^2\phi + V' \right\}\delta\phi \stackrel{!}{=} 0\,.\end{aligned}$$

Thus the field equation for a Klein-Gordon field in a flat FRW background is

$$\ddot\phi + 3H\dot\phi - \frac{1}{a^2}\nabla^2\phi + V' = 0\,.$$

b. Varying the action w.r.t. the metric gives

$$\begin{aligned}\delta S_{\mathrm{KG}} &= \frac{1}{2}\int_\Omega \mathrm{d}^4x \left\{ \sqrt{|g|}\nabla_a\phi\nabla_b\phi\,\delta g^{ab} + [g^{ab}\nabla_a\phi\nabla_b\phi - 2V(\phi)]\delta\sqrt{|g|} \right\} \\ &= \int_\Omega \mathrm{d}^4x\sqrt{|g|}\delta g^{ab}\left\{ \frac{1}{2}\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\mathcal{L} \right\}. \end{aligned} \tag{2}$$

and thus

$$T_{ab} = \frac{2}{\sqrt{|g|}}\frac{\delta S_{\mathrm{m}}}{\delta g^{ab}} = \nabla_a\phi\nabla_b\phi - g_{ab}\mathcal{L}\,. \tag{3}$$

c. Setting the spatial gradients to zero gives

$$T^{ab} = \nabla^a\phi\nabla^b\phi - g^{ab}\mathcal{L} = -\dot\phi^2u^au^b - g^{ab}\left[\frac{1}{2}\dot\phi^2 - V(\phi)\right]$$

with $u^a = (1,0,0,0)$. Comparing with the energy-momentum tensor of a perfect fluid,

$$T^{ab} = (\rho + P)u^au^b + Pg^{ab}$$

we find $P = \dot\phi^2/2 - V(\phi)$ and $\rho = \dot\phi^2/2 + V(\phi)$. Thus

$$w = \frac{P}{\rho} = \frac{\dot\phi^2/2 - V(\phi)}{\dot\phi^2/2 + V(\phi)}$$

d. A necessary condition is $w \to -1$ or $\dot\phi^2 \ll V(\phi)$.

4. Killing vectors.

Consider Minkowski space of special relativity,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,.$$

a. Find all ten Killing vectors and name the conserved symmetries and conserved quantities. (4.5 pt)

a. [It was sufficient to name the symmetries and conservation law, but we give also a short derivation.] The Killing equation $\nabla_i \xi_j + \nabla_j \xi_i = 0$ simplifies in Minkowski space to

$$\partial_i \xi_j = -\partial_j \xi_i \, .$$

The first four obvious Killing vectors are the Cartesian basis vectors,

$$\mathbf{T}_0 = \partial_t, \qquad \mathbf{T}_1 = \partial_x, \qquad \mathbf{T}_2 = \partial_y, \qquad \mathbf{T}_3 = \partial_z,$$

of Minkowski space. Here as in the following, we use for the basis $\mathbf{e}_i$ the notation $\mathbf{e}_i = \partial_i$ (correct because of $\partial_i \mathrm{d}x^j = \delta_i^j$), such that the following equations should remind you to quantum mechanics. In coordinate notation, $\mathbf{T}_0 = (1,0,0,0), \ldots, \mathbf{T}_3 = (0,0,0,1)$. The four Killing vectors $\mathbf{T}_i$ generate translations, $x^i \to x^i + a^i$. For a particle with momentum $\mathbf{p} = m\mathbf{u}$ moving along $\mathbf{x}(\lambda)$, the existence of a Killing vector $\mathbf{T}_i$ implies

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathbf{T}_i \cdot \mathbf{u}) = \frac{\mathrm{d}}{m\mathrm{d}\lambda}(\mathbf{T}_i \cdot \mathbf{p}) = 0$$

i.e. the conservation of the four-momentum component $\mathbf{p}_i$. (Energy for time-like K_i, one component of the three-momentum for space-like K_i.)

Consider next the $\alpha\beta$ (=spatial) components of the Killing equation. Three additional Killing vectors are

$$\begin{aligned} \mathbf{J}_1 &= y\partial_z - z\partial_y \, , \\ \mathbf{J}_2 &= z\partial_x - x\partial_z \, , \\ \mathbf{J}_3 &= x\partial_y - y\partial_x \, . \end{aligned}$$

Remembering QM ($p_i \leftrightarrow \partial_i$), we expect that $\mathbf{J}_i$ generate rotations and that the conserved quantity is the angular momentum $\vec{L}$. Second, we see that we can promote the three components of J_i to anti-symmetric 4-dim. tensor, the other 3 components satisfying the 0α component of the Killing equations ($B_0 = -B^0$),

$$\begin{aligned} \mathbf{B}_1 &= t\partial_z + z\partial_t \, , \\ \mathbf{B}_2 &= t\partial_x + x\partial_t \, , \\ \mathbf{B}_3 &= t\partial_y + y\partial_t \, . \end{aligned}$$

Next we confirm that $\mathbf{J}_i$ generate rotations. We write for an infinitesimal rotation around, e.g. the z axis,

$$\begin{aligned} t' &= t \, , \\ x' &= \cos\alpha x - \sin\alpha y \approx x - \alpha y \, , \\ y' &= \sin\alpha x + \cos\alpha y \approx y + \alpha x \, , \\ z' &= z \, . \end{aligned}$$

Hence $\mathbf{J}_3$ is indeed $\mathbf{J}_3 = (0, -y, x, 0)$. (We could have found ξ_z also by rewriting the line-element in spherical coordinates and noting that ds does not contain ϕ dependent terms ("cyclic coordinate"). The other two rotations follow by cyclic permutation.)
The existence of Killing vectors $\mathbf{J}_i$ implies that $\mathbf{J}_i \cdot \mathbf{p}$ is conserved along a geodesics of particle. But

$$\mathbf{J}_1 \cdot \mathbf{p} = yp_z - zp_y = L_x$$

and thus the angular momentum around the origin of the coordinate system is conserved.
We can repeat the discussion for proper Lorentz transformations (boosts), with the sign changes because of $B_0 = -B^0$ as only difference. Boosts corresponds to rotations in a hyperbolic space (or around an imaginary angle); e.g.

$$\begin{aligned} t' &= \cosh \alpha t + \sinh \alpha x \approx t + \alpha x \,, \\ x' &= \sinh \alpha t + \cosh \alpha x \approx x + \alpha t \,, \\ y' &= y \\ z' &= z \,. \end{aligned}$$

Hence $\mathbf{B}_1 = (t, x, 0, 0)$ and the other two follow again by cyclic permutation. The conserved quantity $tp_z - zE = \text{const.}$ now depends on time and is therefore not as popular... Its conservation implies that the center of mass of a system of particles moves with $v_\alpha = p_\alpha / E$.

5. Radiation from a particle in a gravitational field.
An electron is released at the position $r \gg 2M$ in the gravitational field of a point mass M and moves thereafter on a geodesics. Give either a *short, simple* argument why
a. the electron does not emit radiation. (1 pt)
and
b. the electron does emit radiation. (1 pt)
or
c. decide which one of the alternatives is correct and explain why. (2 pt)

a. We can always find a normal coordinate system at the position of the electron that is locally Minkowskian, transforming gravity away: no gravity, no acceleration, no emitted radiation.
b. For $r \gg 2M$, Newtonian physics is a good approximation: Gravity enters as any other force Newton's law $F = ma$, the electron is accelerated and emits radiation.
c. Argument a. is wrong, because it applies only to the point-like electron but not to its $1/r$ Coulomb field. Tidal effects distort the Coulomb field, leading to the emission of radiation.

Some formula: Signature of the metric $(+, -, -, -)$. [**wrong,** should be $(-, +, +, +)$.

$$T^{ab} = (\rho + P) u^a u^b + P g^{ab}$$

$$\nabla_i \xi_j + \nabla_j \xi_i = 0$$

$$\ddot{x}^c + \Gamma^c{}_{ab}\dot{x}^a\dot{x}^b = 0$$

$$\frac{2}{\sqrt{|g|}}\,\frac{\delta S_{\rm m}}{\delta g^{ab}} = T_{ab}$$

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}\,,$$

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\,g^{ab}\delta g_{ab}$$