NTNU **Institutt for fysikk**

Faglig kontakt under eksamen:
Professor Kåre Olaussen
Telefon: 45 43 71 70

Eksamen i FY3452 GRAVITASJON OG KOSMOLOGI

Lørdag 11. august 2012
09:00–13:00

Tillatte hjelpemidler: Alternativ **C**
Standard kalkulator (ifølge NTNU's liste).
K. Rottman: *Matematisk formelsamling* (alle språkutgaver).
Barnett & Cronin: *Mathematical Formulae*

There is also an english version of this exam set.

Dette oppgavesettet er på 3 sider.

Oppgave 1. Raskere-enn-lyset (superluminal) bevegelse?
Sammenligning av observerte kjente spektrallinjer fra kvasaren 3C345 (med bølgelengde λ_{3C345}) med tilsvarende fra jorda (med bølgelengde λ_{earth}) viser en betydelig rødforskyvning:

$$\frac{\lambda_{3C345}}{\lambda_{\text{earth}}} \equiv 1 + z \approx 1.595. \tag{1}$$

a) Anta først at observasjonen (1) i sin helhet skyldes universets ekspansjon, og bruk den til å anslå avstanden fra jorda til 3C345 (i) da strålingen ble emittert, og (ii) idag. Begge deler målt i kosmologisk medfølgende (comoving) koordinater, dvs koordinatsystemet der den kosmiske bakgrunnsstrålingen er isotrop.

b) Anta istedet at observasjonen (1) i sin helhet skyldes at jorda og 3C345 beveger seg fra hverandre i et flatt tidrom (Minkowski-rommet) med hastighet v, og bruk den til å bestemme v.

c) Den største anisotropien som man måler i den kosmiske bakgrunnsstrålingen kan forklares ved at jorda beveger seg i forhold til det medfølgende koordinatsystemet (der den kosmiske bakgrunnsstrålingen er isotrop). Med omtrent hvilken hastighet, v_{earth}, tror du denne bevegelsen skjer idag?

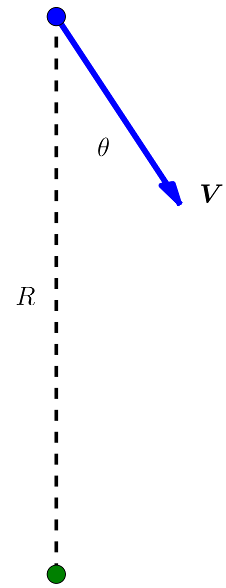
d) I virkeligheten må man forvente at rødforskyvningen (1) både skyldes universets ekspansjon, og at 3C345 og jorda har hver sine hastigheter målt i medfølgende koordinater. Anslå hvor stor usikkerhet dette siste gir i avstandene du fant i punkt **a)**.

e) Observasjoner av skyer i 3C345 har identifisert en sky som beveger seg med en vinkelhastighet

$$\omega = 2.3 \times 10^{-9}\,\frac{\text{radianer}}{\text{år}} \tag{2}$$

i forhold til sentrum av 3C345. Anta at denne skyen beveger seg på tvers av synsretningen. Med hvilken hastighet $V_\perp$ må den da bevege seg?

f) Med riktig regning skal du ha funnet at $V_\perp \gg c$ i forrige punkt(!). Dette resultatet har vært brukt til å argumentere for at kvasarer umulig kan være så langt borte som rødforskyningen indikerer. Det er imidlertid en annen mulig forklaring, som du skal utforske her.
Anta at skyen ikke beveger seg på tvers av synsretningen, men med en vinkel θ i forhold til den, som skissert i figuren til høyre. Den hastigheten $|\boldsymbol{V}|$ som kreves for å forklare observasjonen (2) vil da avhenge av hvilken vinkel θ vi antar. Finn denne sammenhengen. Hva er den minste verdien $|\boldsymbol{V}|$ kan ha?



Tips 1: For å forenkle analysen kan du anta at alt skjer i Minkowski-rommet.

Tips 2: Husk å ta hensyn til den tiden det tar fra strålingen emitteres til den detekteres.

Oppgitt: Hubble "konstanten" er idag (tid t_0)

$$H_0 \equiv \frac{d}{dt} \log(a(t))\bigg|_{t=t_0} \approx \frac{0.721}{9.777\,752 \times 10^{10} \text{ år}}. \tag{3}$$

Oppgave 2. Ladet skalfelt

Det komplekse Klein-Gordon feltet kan brukes til å modellere en samling av ladete skalar-partikler. Dynamikken er definert av Langrange-funksjonen

$$\mathcal{L} = \eta^{\mu\nu} \left(D_\mu \varphi\right)^* D_\nu \varphi - \kappa^2 \varphi^* \varphi - \lambda \left(\varphi^* \varphi\right)^2 \tag{4}$$

der $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$ er den kovariant deriverte for elektrodynamikk, $|\kappa|^{-1}$ er en karakteristisk lengde-parameter, og λ er en vekselvirkningsparameter. Vi bruker metrikken $\eta^{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$.

a) Hva er Euler-Lagrange ligningen for φ?

b) Hva er Euler-Lagrange ligningen for φ^*?

c) Vis at Lagrange-funksjonen (4) er invariant under globale fasetransformasjoner,

$$\varphi(x) \to \varphi'(x) = \mathrm{e}^{\mathrm{i}\alpha}\, \varphi(x), \quad \varphi^*(x) \to \varphi^{*\prime}(x) = \mathrm{e}^{-\mathrm{i}\alpha}\, \varphi^*(x). \tag{5}$$

Hva er de infinitesemale versjonene av disse transformasjonene?

d) Hva er den konserverte Nöther-strømmen som følger av denne invariansen?

e) Hva er de kanonisk konjugerte impulstetthetene Π_φ og Π_{φ^*} til henholdsvis feltet φ og φ^*?

f) Hva er Hamilton-tettheten $\mathcal{H}$ (energitettheten) for denne modellen?

g) Anta at $A_0 \neq 0$ (men konstant) og $\boldsymbol{A} = 0$. Hva er laveste energi for dette systemet? Hvordan avhenger resultatet av verdien κ^2 (som du kan anta å være både positiv og negativ)?

Oppgave 3. Indekser, indekser . . .

Mange rutineberegninger i generell relativitetsteori går ut på å manipulere indekser. Tabellen under viser noen resultater av slike manipulasjoner.

Hvilke av disse er (**R**) opplagt riktige, (**W**) opplagt gale, og (**M**) kanskje riktige—kanskje gale (avhengig av størrelsene som er involvert)?

a)	$g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta}\, dx^\alpha dx^\gamma$	
b)	$g_{\alpha\beta}\, a^\alpha b^\beta = g_{\alpha\beta}\, a^\alpha c^\beta$	
c)	$g_{\alpha\beta}\, a^\alpha b^\beta = g_{\beta\gamma}\, a^\gamma b^\beta$	
d)	$\Gamma^\alpha{}_{\alpha\gamma}\, a^\gamma = g_{\alpha\beta}\, a^\alpha b^\beta$	
e)	$\Gamma^\alpha{}_{\beta\gamma}\, a^\alpha b^\beta c^\gamma = b^\alpha$	
f)	$\partial x^\alpha / \partial x^\beta = \delta^\alpha_\beta$	
g)	$\partial g_{\alpha\beta} / \partial x^\gamma = 0$	
h)	$g_{\alpha\beta} \dfrac{\partial x^\alpha}{\partial x'^\gamma} \dfrac{\partial x^\beta}{\partial x'^\delta} = g_{\gamma\delta} \dfrac{\partial x^\gamma}{\partial x'^\alpha} \dfrac{\partial x^\delta}{\partial x'^\beta}$	
i)	$g'_{\alpha\beta}\, a'^\alpha b'^\beta = g_{\alpha\beta}\, a^\alpha b^\beta$	
j)	$a^\alpha \left(g_{\beta\gamma} b^\beta b^\gamma \right) = b^\gamma$	
k)	$\Gamma^\alpha{}_{\alpha\beta} = \Gamma^\beta{}_{\beta\beta}$	
l)	$g_{\alpha\beta} = \eta_{\alpha\beta}$	
m)	$\xi^\mu \partial_\mu = \xi_\rho \partial^\rho$	
n)	$g_{\mu\rho}\, \xi^\rho{}_{,\nu} = \xi_{\mu,\nu}$	
o)	$g_{\mu\rho}\, \xi^\rho{}_{;\nu} = \xi_{\mu;\nu}$	

Some expressions which *may* be of use

Euler-Lagrange equations

The Euler-Lagrange equations for a field theory described by the Lagrangian $\mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu \varphi_a, x)$ are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi_a}. \tag{6}$$

The corresponding equations for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.

Nöther's theorem

Assume the action is invariant under the continuous transformations $\varphi_a \to \varphi_a + \varepsilon\, \delta\varphi_a + \mathcal{O}(\varepsilon^2)$, more precisely that $\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\mu \Lambda^\mu + \mathcal{O}(\epsilon^2)$ under this transformation. Then there is an associated conserved current,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a - \Lambda^\mu. \tag{7}$$

I.e., $\partial_\mu J^\mu = 0$. The corresponding expression for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.

NTNU **Institutt for fysikk**

Contact during the exam:
Professor Kåre Olaussen
Telephone: 45 43 71 70

Exam in FY3452 GRAVITATION AND COSMOLOGY

Saturday august 11, 2012
09:00–13:00

Allowed help: Alternativ **C**
Standard calculator (according to list made by NTNU).
K. Rottman: *Matematisk formelsamling* (all languages).
Barnett & Cronin: *Mathematical Formulae*

Det er også en norsk versjon av dette eksamenssettet.

This problemset consists of 3 pages.

Problem 1. Faster-than-light (superluminal) motion?
Comparison of observed known spectral lines from the quasar 3C345 (with wavelength λ_{3C345}) with corresponding ones from earth (with wavelength λ_{earth}) show a considerable redshift

$$\frac{\lambda_{3C345}}{\lambda_{\text{earth}}} \equiv 1 + z \approx 1.595. \tag{1}$$

a) First assume that the observation (1) is entirely due to the expansion of the universe, and use it to estimate the distance between earth and 3C345 (i) when the radiation was emitted, and (ii) today. Both measured in cosmological comoving coordinates, i.e. a coordinate system where the cosmic background radiation is isotropic.

b) Next assume instead that the observation (1) is entirely due to a relative motion between earth and 3C345 in flat spactime (Minkowski space, with relative velocity v, and use it to determine v.

c) The largest anisotropy measured in the cosmic background radiation kan be explained by the earth's motion relative to the comoving coordinate system (where the cosmic background ratiation is isotropic). About which velocity, v_{earth}, do you think this motion has today?

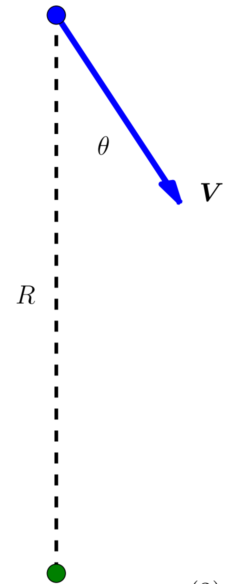
d) In the real world one must expect the redshift (1) to be due to both the expansion of the universe, and the motions of 3C345 and earth in comoving coordinates. Estimate which uncertainty the latter motions have on the distances you found in point **a)**.

e) Observations of clouds in 3C345 have identified a cloud moving with angular velocity

$$\omega = 2.3 \times 10^{-9}\,\frac{\text{radians}}{\text{year}} \tag{2}$$

relative to the center of 3C345. Assume that this cloud moves in a direction orthogonal to the line of sight. With which velocity $V_\perp$ must it move?

f) If you calculated correctly you should have found $V_\perp \gg c$ in the previous point(!). This result has been used to argue that quasars cannot possibly be so far away as their redshift indicates. There is however another possible explanation, which you should investigate here.
Assume that the cloud move at an angle θ relative to the line of sight, as indicated in the figure to the right. The velocity $|\boldsymbol{V}|$ required to explain the observation (2) will depend on which angle θ we assume. Find this connection. What is the lowest value $|\boldsymbol{V}|$ may have?



Hint 1: To simplify the analysis you may assume that everything happens in Minkowski space.

Hint 2: Remember to take into accound the time from the radiation is emitted until it is detected.

Oppgitt: The Hubble "constant" today (time t_0) is

$$H_0 \equiv \frac{d}{dt} \log(a(t))\Big|_{t=t_0} \approx \frac{0.721}{9.777\,752 \times 10^{10}\ \text{year}}. \tag{3}$$

Problem 2. Charged scalar field
The complex Klein-Gordon field can be used to model a collection of charged scalar particles. The dynamics is defined by the Langrangian

$$\mathcal{L} = \eta^{\mu\nu} \left(D_\mu \varphi\right)^* D_\nu \varphi - \kappa^2 \varphi^* \varphi - \lambda \left(\varphi^* \varphi\right)^2 \tag{4}$$

where $D_\mu = \partial_\mu + \mathrm{i} q A_\mu$ is the covariant derivative of electrodynamics, $|\kappa|^{-1}$ is a characteristic length parameter, and λ is an interaction parameter. We use the metric $\eta^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$.

a) What is the Euler-Lagrange equation for φ?

b) What is the Euler-Lagrange equation for φ^*?

c) Show that the Lagrangian (4) is invariant under global phase transformations,

$$\varphi(x) \to \varphi'(x) = \mathrm{e}^{\mathrm{i}\alpha}\, \varphi(x), \quad \varphi^*(x) \to \varphi^{*\prime}(x) = \mathrm{e}^{-\mathrm{i}\alpha}\, \varphi^*(x). \tag{5}$$

What are the infinitesemal versions of these transformations?

d) What is the conserved Nöther current resulting from this invariance?

e) What are the canonically conjugate momentum densities Π_φ and Π_{φ^*} of respectively the field φ and φ^*?

f) What is the Hamiltonian density $\mathcal{H}$ (energy density) of this model?

g) Assume that $A_0 \neq 0$ (but constant) and $\boldsymbol{A} = 0$. What is the lowest energy of this system? How does this result depend on the value of A_0?

Problem 3. Indices, indices . . .

Many routine computations in General Relativity consists of index manipulations. The table below show some results of such manipulations. Which of these are (**R**) obviously right, (**W**) obviously wrong, and (**M**) maybe right—maybe wrong (depending on the quantities involved)?

a)	$g_{\alpha\beta}\,\mathrm{d}x^\alpha \mathrm{d}x^\beta = g_{\alpha\beta}\,\mathrm{d}x^\alpha \mathrm{d}x^\gamma$	
b)	$g_{\alpha\beta}\,a^\alpha b^\beta = g_{\alpha\beta}\,a^\alpha c^\beta$	
c)	$g_{\alpha\beta}\,a^\alpha b^\beta = g_{\beta\gamma}\,a^\gamma b^\beta$	
d)	$\Gamma^\alpha{}_{\alpha\gamma}\,a^\gamma = g_{\alpha\beta}\,a^\alpha b^\beta$	
e)	$\Gamma^\alpha{}_{\beta\gamma}\,a^\alpha b^\beta c^\gamma = b^\alpha$	
f)	$\partial x^\alpha / \partial x^\beta = \delta^\alpha_\beta$	
g)	$\partial g_{\alpha\beta} / \partial x^\gamma = 0$	
h)	$g_{\alpha\beta}\,\dfrac{\partial x^\alpha}{\partial x'^\gamma}\dfrac{\partial x^\beta}{\partial x'^\delta} = g_{\gamma\delta}\,\dfrac{\partial x^\gamma}{\partial x'^\alpha}\dfrac{\partial x^\delta}{\partial x'^\beta}$	
i)	$g'_{\alpha\beta}\,a'^\alpha b'^\beta = g_{\alpha\beta}\,a^\alpha b^\beta$	
j)	$a^\alpha\left(g_{\beta\gamma} b^\beta b^\gamma\right) = b^\gamma$	
k)	$\Gamma^\alpha{}_{\alpha\beta} = \Gamma^\beta{}_{\beta\beta}$	
l)	$g_{\alpha\beta} = \eta_{\alpha\beta}$	
m)	$\xi^\mu \partial_\mu = \xi_\rho \partial^\rho$	
n)	$g_{\mu\rho}\,\xi^\rho{}_{,\nu} = \xi_{\mu,\nu}$	
o)	$g_{\mu\rho}\,\xi^\rho{}_{;\nu} = \xi_{\mu;\nu}$	

Some expressions which *may* be of use

Euler-Lagrange equations

The Euler-Lagrange equations for a field theory described by the Lagrangian $\mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu \varphi_a, x)$ are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi_a}. \tag{6}$$

The corresponding equations for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.

Nöther's theorem

Assume the action is invariant under the continuous transformations $\varphi_a \to \varphi_a + \varepsilon\, \delta\varphi_a + \mathcal{O}(\varepsilon^2)$, more precisely that $\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\mu \Lambda^\mu + \mathcal{O}(\epsilon^2)$ under this transformation. Then there is an associated conserved current,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a - \Lambda^\mu. \tag{7}$$

I.e., $\partial_\mu J^\mu = 0$. The corresponding expression for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.