NTNU **Institutt for fysikk**

Faglig kontakt under eksamen:
Professor Kåre Olaussen
Telefon: 45 43 71 70

Eksamen i FY3452 GRAVITASJON OG KOSMOLOGI

Fredag 24. mai 2013
09:00–13:00

Tillatte hjelpemidler: Alternativ **C**
Standard kalkulator (ifølge NTNU's liste).
K. Rottman: *Matematisk formelsamling* (alle språkutgaver).
Barnett & Cronin: *Mathematical Formulae*

There is also an english version of this exam set.

Dette oppgavesettet er på 2 sider.

Oppgave 1. Bevegelse utenfor et roterende legeme

Til laveste ikke-trivielle orden i r_M/r er linjeelementet utenfor et roterende legeme med masse M og dreieimpuls J av formen

$$c^2\,\mathrm{d}\tau^2 = \left(1-\frac{r_M}{r}\right)c^2\mathrm{d}t^2 + 2K_J\frac{r_M^2}{r^2}\sin^2\theta\, cr\,\mathrm{d}t\,\mathrm{d}\phi - \left(1+\frac{r_M}{r}\right)\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right). \quad (1)$$

Her er

$$r_M = \frac{2G_N M}{c^2},\quad K_J = \frac{J}{Mc\,r_M}, \quad (2)$$

der G_N er Newton's gravitasjonskonstant. Bevegelsen til en punktpartikkel utenfor dette legemet er bestemt av Lagrangefunksjonen

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\,\dot{x}^\nu, \quad (3)$$

via Hamiltons prinsipp. Her betyr $\dot{}$ derivasjon med hensyn til egentid τ. Du kan velge å bruke enheter der $c=1$.

a) Lagrangefunksjonen L avhenger ikke eksplisitt av t. Hvilken konservert størrelse gir dette opphav til?

b) Lagrangefunksjonen L avhenger ikke eksplisitt av ϕ. Hvilken konservert størrelse gir dette opphav til?

c) Lagrangefunksjonen L avhenger ikke eksplisitt av τ. Hvilken konservert størrelse gir dette opphav til?

d) Anta at $\theta=\frac{1}{2}\pi$, $\dot{\theta}=0$, dvs. bevegelse i ekvatorplanet, er en løsning av bevegelsesligningene. Sett derfor $\sin^2\theta=1$, $\dot{\theta}=0$, og finn bevegelsesligningen for $r(\tau)$.

Oppgave 2. Estimat av størrelsesorden

Bruk din generelle kunnskap om fysiske fenomener og fysiske sammenhenger til å anslå størrelsene nedenfor. Forklar hvordan du kom fram til anslagene.

a) Parameteren $r_{M_\oplus}/r_\oplus$, der $M_\oplus$ er massen til jorda og $r_\oplus$ er jordas radius.

b) Parameteren $r_{M_\odot}/r_\odot$, der $M_\odot$ er massen til sola og $r_\odot$ er solas radius.

c) Parameteren $K_{J_\oplus} = \frac{J_\oplus}{M_\oplus c\, r_\oplus}$ for jorda, der $J_\oplus$ er dreieimpulsen til jorda.

d) Parameteren $K_{J_\odot} = \frac{J_\odot}{M_\odot c\, r_\odot}$ for sola, der $J_\odot$ er dreieimpulsen til sola.

Oppgave 3. Einstein's gravitasjonsteori til laveste orden

I denne oppgaven skal du se litt på Einstein gravitasjonsteori til første orden i avviket fra flatt rom. Dvs. at vi skriver linjeelementet på formen

$$c^2\,\mathrm{d}\tau^2 = \{\eta_{\mu\nu} + \varepsilon\, h_{\mu\nu}(x)\}\,\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{4}$$

der $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$, og bare regner til første orden i parameteren ε. Dette er tilstrekkelig til å relativt enkelt kunne finne linjeelementer som f.eks. det i ligning (1).

a) Anta at vi gjør en (liten) transformasjon av koordinater,

$$x^\mu = \tilde{x}^\mu + \varepsilon\,\Lambda^\mu(\tilde{x}), \tag{5}$$

og regn ut den tilhørende transformasjonen,

$$h_{\mu\nu}(x) \to \tilde{h}_{\mu\nu}(\tilde{x}). \tag{6}$$

b) Vis at det er mulig å velge $\Lambda^\mu(\tilde{x})$ slik at

$$V_\nu(\tilde{h}) \equiv \partial_\mu \left(\tilde{h}^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu\,\tilde{h}^\lambda{}_\lambda\right) = 0. \tag{7}$$

I det følgende kan du anta at denne betingelsen allerede er oppfylt for $h_{\mu\nu}$, dvs. at $V_\nu(h) = 0$.

c) Bestem konneksjonskoeffisientene $\Gamma^\mu{}_{\nu\lambda}$ til første orden i ε.

d) Vis at Riemann-tensoren kan uttrykkes på formen

$$R_{\mu\nu\lambda\sigma} = \frac{1}{2}\left(h_{\mu\sigma,\nu\lambda} + h_{\nu\lambda,\mu\sigma} - h_{\nu\sigma\mu\lambda} - h_{\mu\lambda,\nu\sigma}\right)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{8}$$

e) Hver av de fire indeksene til $R_{\mu\nu\lambda\sigma}$ kan ta fire verdier $(0, 1, 2, 3)$. Hvor mange *uavhengige* komponenter har $R_{\mu\nu\lambda\sigma}$ for en generell symmetrisk $h_{\mu\nu}$?

f) Beregn Ricci-tensoren $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$. Bruk betingelsen $V_\nu(h) = 0$ til å forenkle uttrykket. Beregn Einstein-tensoren $G_{\mu\nu} = R_{\mu\nu} - \eta_{\mu\nu}\,R^\lambda{}_\lambda$ under den samme betingelsen.

Some expressions which *may* be of use

Euler-Lagrange equations

The Euler-Lagrange equations for a field theory described by the Lagrangian $\mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu \varphi_a, x)$ are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi_a}. \tag{9}$$

The corresponding equations for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.

Nöther's theorem

Assume the action is invariant under the continuous transformations $\varphi_a \to \varphi_a + \varepsilon\, \delta\varphi_a + \mathcal{O}(\varepsilon^2)$, more precisely that $\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\mu \Lambda^\mu + \mathcal{O}(\epsilon^2)$ under this transformation. Then there is an associated conserved current,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a - \Lambda^\mu. \tag{10}$$

I.e., $\partial_\mu J^\mu = 0$. The corresponding expression for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.

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Contact during the exam:
Professor Kåre Olaussen
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Exam in FY3452 GRAVITATION AND COSMOLOGY

Friday May 24, 2013
09:00–13:00

Allowed help: Alternativ **C**
Standard calculator (according to list by NTNU).
K. Rottman: *Matematisk formelsamling* (all language editions).
Barnett & Cronin: *Mathematical Formulae*

Det er også en norsk versjon av dette eksamenssettet.

This problemset consists of 2 pages.

Problem 1. Motion outside a rotating body

The line element outside a rotating body with mass M and angular momentum J is to lowest non-trivial order in r_M/r of the form

$$c^2\,\mathrm{d}\tau^2 = \left(1-\frac{r_M}{r}\right)c^2\mathrm{d}t^2 + 2K_J\frac{r_M^2}{r^2}\sin^2\theta\,cr\,\mathrm{d}t\,\mathrm{d}\phi - \left(1+\frac{r_M}{r}\right)\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right). \quad (1)$$

Here

$$r_M = \frac{2G_N M}{c^2}, \quad K_J = \frac{J}{Mc\,r_M}, \quad (2)$$

where G_N is the Newton constant of gravity. The motion of a point particle outside this body is governed by the Lagrange function,

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\,\dot{x}^\nu, \quad (3)$$

through Hamilton's principle. Here $\dot{}$ means differentiation with respect to eigentime τ. You may choose to use units where $c=1$.

a) The Lagrange function L does not depend explicitly on t. Which conserved quantity does this give rise to?

b) The Lagrange function L does not depend explicitly on ϕ. Which conserved quantity does this give rise to?

c) The Lagrange function L does not depend explicitly on τ. Which conserved quantity does this give rise to?

d) Assume that $\theta = \frac{1}{2}\pi$, $\dot{\theta}=0$, i.e. motion in the equatorial plane, is a solution of the equations of motion. Thus set $\sin^2\theta = 1$, $\dot{\theta}=0$, and find the equation of motion for $r(\tau)$.

Problem 2. Estimating orders of magnitude

Use your general knowledge of physical phenomena and physical relations to estimate the quantities below. Explain how you arrive at the estimates.

a) The parameter $r_{M_\oplus}/r_\oplus$, where $M_\oplus$ is the mass of the earth, and $r_\oplus$ is the radius of the earth.

b) The parameter $r_{M_\odot}/r_\odot$, where $M_\odot$ is the mass of the sun, and $r_\odot$ is the radius of the sun.

c) The parameter $K_{J_\oplus} = \frac{J_\oplus}{M_\oplus c\, r_\oplus}$ for the earth, where $J_\oplus$ is the angular momentum of the earth.

d) The parameter $K_{J_\odot} = \frac{J_\odot}{M_\odot c\, r_\odot}$ for the sun, where $J_\odot$ is the angular momentum of the sun.

Problem 3. Einstein gravity to lowest order

In this problem you shall investigate the Einstein theory of gravity to first order in the deviation from flat space. I.e., we write the line element in the form

$$c^2\, \mathrm{d}\tau^2 = \{\eta_{\mu\nu} + \varepsilon\, h_{\mu\nu}(x)\}\, \mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{4}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$, and calculate only to first order in the parameter ε. This is sufficient to relatively easy be able to determine line elements like f.i. the one in equation (1).

a) Assume we make a (small) transformation of coordinates,

$$x^\mu = \tilde{x}^\mu + \varepsilon\, \Lambda^\mu(\tilde{x}), \tag{5}$$

and calculate the corresponding transformation,

$$h_{\mu\nu}(x) \to \tilde{h}_{\mu\nu}(\tilde{x}). \tag{6}$$

b) Show that it is possible to choose $\Lambda^\mu(\tilde{x})$ such that

$$V_\nu(\tilde{h}) \equiv \partial_\mu \left(\tilde{h}^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu\, \tilde{h}^\lambda{}_\lambda \right) = 0. \tag{7}$$

In the following you may assume that this condition is already fulfilled for $h_{\mu\nu}$, i.e. that $V_\nu(h) = 0$.

c) Determine the connection coefficients $\Gamma^\mu{}_{\nu\lambda}$ to first order in ε.

d) Show that the Riemann tensor can be expressed in the form

$$R_{\mu\nu\lambda\sigma} = \frac{1}{2}\left(h_{\mu\sigma,\nu\lambda} + h_{\nu\lambda,\mu\sigma} - h_{\nu\sigma\mu\lambda} - h_{\mu\lambda,\nu\sigma}\right)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{8}$$

e) Each of the four indices of $R_{\mu\nu\lambda\sigma}$ may take four values $(0,1,2,3)$. How many *independent* components does $R_{\mu\nu\lambda\sigma}$ have for a general symmetric $h_{\mu\nu}$?

f) Calculate the Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$. Use the condition $V_\nu(h) = 0$ to simplify the expression. Calculate the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \eta_{\mu\nu}\, R^\lambda{}_\lambda$ under the same condition.

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I.e., $\partial_\mu J^\mu = 0$. The corresponding expression for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.