NTNU – Trondheim
Norwegian University of
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# Exam FY3452 Gravitation and Cosmology fall 2016

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09.00-13.00

Permitted examination support material:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of four pages. Read carefully. Good luck! Bonne chance! Viel Glück! Veel succes! Lykke til!

## Problem 1

Consider the standard situation where an inertial frame $S'$ moves along the positive $x$-axis with speed $v$ relative to another inertial frame $S$.

**a)** Show the relation between the acceleration $a'_x$ in $S'$ and $a_x$ in $S$

$$a'_x = \frac{1}{\gamma}\frac{a_x}{(1-\frac{vV_x}{c^2})^2} + \frac{1}{\gamma}\frac{V_x - v}{(1-\frac{vV_x}{c^2})^3}\frac{va_x}{c^2} \tag{1}$$

Show that if $S'$ is the instantaneous rest frame of a particle moving along the $x$-axis in $S$, $a'_x$ reduces to

$$a'_x = \gamma^3 a_x . \tag{2}$$

**b)** Assume that the acceleration of the particle moving along the $x$-axis is constant in its instantaneous rest frame and equal to $a'_x = g$. Show that $V_x(t) = \frac{dx}{dt}$ is

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}} , \tag{3}$$

if the initial condition is $V_x(0) = 0$. What is the limiting velocity $V_{\text{lim}}$ of $V_x(t)$ as $t \to \infty$?

**c)** The time $t$ in $S$ can be expressed as a function of the proper time $\tau$ of the particle. Show that

$$t(\tau) = \frac{c}{g}\sinh(\tfrac{g}{c}\tau) , \tag{4}$$

if the initial condition is $t(0) = 0$.

**d)** The position $x$ in $S$ can be expressed as a function of the proper time $\tau$ of the particle. Show that

$$x(\tau) = \frac{c^2}{g}\left[\cosh(\tfrac{g}{c}\tau) - 1\right] , \tag{5}$$

if the initial condition is $x(0) = 0$.

**e)** The functions $t(\tau)$ and $x(\tau)$ are the time and position of the origin of $S'$ in $S$. We next consider an arbitrary point in spacetime, whose coordinates in $S'$ are $x'$ and $t' = \tau$. The coordinates of this point in $S$ are given by

$$t = \left[\frac{c}{g} + \frac{x'}{c}\right]\sinh(\tfrac{g}{c}\tau) , \tag{6}$$

$$x = \frac{c^2}{g}\left[\cosh(\tfrac{g}{c}\tau) - 1\right] + x'\cosh(\tfrac{g}{c}\tau) . \tag{7}$$

Show that the metric can be written as

$$ds^2 = -c^2dt'^2\left(1+\frac{gx'}{c^2}\right)^2 + dx'^2 + dy'^2 + dz'^2 . \tag{8}$$

**f)** Explain why $\xi = (1, 0, 0, 0)$ is a Killing vector and find the associated conserved quantity.

**g)** Calculate the redshift of a photon that is emitted at $x' = h$ and absorbed at $x' = 0$. Explain the result.

## Problem 2

The two Friedman equations are given by

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi\rho + \Lambda\,, \tag{9}$$

$$\frac{2\ddot{a}a + \dot{a}^2 + k}{a^2} = -8\pi p + \Lambda\,, \tag{10}$$

where $a(t)$ is the scale factor, $k = 0, \pm 1$ is the spatial curvature, $\rho$ is the energy density of matter and radiation, $p$ is the pressure, and $\Lambda > 0$ is the cosmological constant.

**a)** In Einstein's static model for the universe, there is no radiation present ($\rho = \rho_m$) and the pressure vanishes. Moreover the spatial curvature is positive, $k = +1$. Show that

$$\ddot{a} = -\frac{4\pi}{3}a\rho_m + \frac{1}{3}a\Lambda\,. \tag{11}$$

**b)** For a given value of $\Lambda$, there is a critical value of $\rho_m$, $\rho_m^c$, such that $a$ is time independent. Find the value of $\rho_m^c$ in terms of $\Lambda$. Find the corresponding value of $a = a_c$ in terms of $\Lambda$.

**c)** We will next study the stability of the static universe. We consider a small perturbation $\delta\rho_m$ of the density around $\rho_m^c$ and write $\rho_m = \rho_m^c + \delta\rho_m$. we can then write $a = a + \delta a$, where $\delta a$ is the corresponding change in the scale factor. $\delta\rho_m$ and $\delta a$ are time dependent. Use the Friedman equations to show that $\delta a$ satisfies the second-order differential equation

$$\frac{d^2\delta a}{dt^2} = B\delta a\,, \tag{12}$$

where $B$ is a constant. Calculate $B$. Use this result to determine whether Einstein's static universe is stable or unstable. (Help: Even if you cannot find $B$, you can still say something about the stability).

Useful formulas

$$
\begin{aligned}
x' &= \gamma(x - vt)\ , && (13)\\
t' &= \gamma\left(t - \frac{v}{c^2}x\right)\ , && (14)\\
V'_x &= \frac{V_x - v}{1 - \frac{vV_x}{c^2}}\ . && (15)
\end{aligned}
$$