NTNU – Trondheim
Norwegian University of
Science and Technology

Exam FY3452 Gravitation and Cosmology fall 2017

Lecturer: Professor Jens O. Andersen
Department of Physics, NTNU
Phone: 46478747 (mob)

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09.00-13.00

Permitted examination support material:
Rottmann: Matematisk Formelsamling
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Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of four pages. We use units such that $c = G = 1$. The metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Read carefully. Good luck! Bonne chance! Viel Glück! Veel succes! Lykke til!

Problem 1

Consider a three-dimensional spacetime with the following metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\phi^2 \ , \tag{1}$$

where (r, ϕ) are polar coordinates in two dimensions and $f(r)$ is an unknown smooth function.

a) Write down the nonzero components $g_{\alpha\beta}(r, \phi)$ of the metric. Is the metric diagonal? Find two Killing vectors and the associated conserved quantities. What is the interpretation of these quantities?

b) Calculate the Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$.

c) Calculate the diagonal components of the Ricci tensor $R_{\alpha\beta}$. Calculate the Ricci scalar R.

d) Determine the function $f(r)$ for the case $T_{\mu\nu} = 0$ by solving Einstein's field equations in vacuum. Any comments?

Problem 2

Flat spacetime has the metric

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dt^2 + dr^2 + r^2 d\phi^2 + dz^2 , \end{aligned} \tag{2}$$

where the second line is expressed in cylindrical coordinates. Consider this line element in a frame that is rotating with an angular velocity $\mathbf{\Omega} = \Omega \mathbf{e}_z$.

a) Show that the metric can be written as

$$ds^2 = -[1 - \Omega^2(x^2 + y^2)]dt^2 + 2\Omega(y dx - x dy)dt + dx^2 + dy^2 + dz^2 . \tag{3}$$

b) The geodesic equations for x, y, and z can be written as

$$\frac{d^2x}{d\tau^2} - 2\Omega \frac{dy}{d\tau}\frac{dt}{d\tau} - \Omega^2 x \left(\frac{dt}{d\tau}\right)^2 = 0 , \tag{4}$$

$$\frac{d^2y}{d\tau^2} + 2\Omega \frac{dx}{d\tau}\frac{dt}{d\tau} - \Omega^2 y \left(\frac{dt}{d\tau}\right)^2 = 0 , \tag{5}$$

$$\frac{d^2z}{d\tau^2} = 0 . \tag{6}$$

Explain why these equations in the nonrelativistic limit reduce to

$$\frac{d^2x}{dt^2} - 2\Omega\frac{dy}{dt} - \Omega^2 x = 0 \,, \qquad (7)$$

$$\frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} - \Omega^2 y = 0 \,, \qquad (8)$$

$$\frac{d^2z}{d^2t} = 0 \,. \qquad (9)$$

c) Write Eqs. (7)–(9) in vector form and interpret the different terms.

Problem 3

a) The covariant derivative ∇_α of a contravariant vector A^β is defined by

$$\nabla_\alpha A^\beta = \partial_\alpha A^\beta + \Gamma^\beta_{\alpha\gamma} A^\gamma \,. \qquad (10)$$

Use Eq. (10) to derive how the covariant derivative ∇_α of a covariant vector A_β should be defined in a consistent manner. Hint: Leibniz' rule.

b) Explain briefly gravitational redshift.

c) Explain briefly the idea of homogeneous and isotropic universe models.

d) The Lagrangian for the electromagnetic field with sources is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \,, \qquad (11)$$

where $j_\mu = (-\rho, \vec{j})$ is the four-current. Show that the Lagrangian is not gauge invariant. Is this a problem? Explain!

Problem 4

Consider a spaceship hovering around a black hole at a radius $R \geq 2M$, where r_s is the Schwarzschild radius and M is its mass. The rest mass of the spaceship is initially m. The commander plans to propel the spaceship to infinity by ejecting part of the rest mass. The material is ejected with the speed of light. What is the fraction f of the rest mass of the spaceship that can escape to infinity? Find the limit f_{horizon} of f as R approaches $2M$.

Useful formulas

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left[\frac{\partial g_{\alpha\delta}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\delta}} \right] , \tag{12}$$

$$R_{\alpha\beta} = \partial_{\gamma}\Gamma^{\gamma}_{\alpha\beta} - \partial_{\beta}\Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta} - \Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} . \tag{13}$$