NTNU – Trondheim
Norwegian University of
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# Exam FY3452 Gravitation and Cosmology summer 2018

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09.00-13.00

Permitted examination support material:
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of four pages. We use units such that $c = G = 1$. The metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Read carefully. Good luck!

## Problem 1

Consider a four-dimensional spacetime whose geometry is described by the line element

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \ , \tag{1}$$

where $M$ is a positive constant, and $r$, $\theta$, and $\phi$ are spherical coordinates.

**a)** Classify the singularities in the geometry described by the line element (1). No proof is required.

**b)** Find two symmetries of the line element (1). Write down the corresponding Killing vectors $\xi$ and $\eta$, and the conserved quantities $e$ and $l$. Give a physical interpretation of $e$ and $l$.

**c)** A massive particle is moving in the spacetime described by the line element (1). Use the results in **b)** to derive an equation of the form

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \;=\; \frac{1}{2}\left(e^2 - 1\right) , \tag{2}$$

where $V_{\text{eff}}(r)$ is a socalled effective potential.

**d)** A massive particle starts at rest at $r = \infty$ falling radially inwards. Show that the smallest radius is $r_{\text{min}} = \frac{1}{2}M$. Calculate the proper time $\Delta\tau$ it takes for the particle to travel from $r = M$ to $r_{\text{min}} = \frac{1}{2}M$.

**e)** We introduce a new time coordinate $\tilde{t}$ via

$$d\tilde{t} \;=\; dt - dr + \frac{dr}{\left(1 - \frac{M}{r}\right)^2} . \tag{3}$$

Find the differential equations for the light-like curves in the spacetime specified by the line element (1). Sketch the worldlines of light in the $(r, \tilde{t})$-plane. Is this the geometry of a black hole?

**f)** We introduce a new variable $v$ via

$$dv \;=\; d\tilde{t} + dr . \tag{4}$$

Express the line element in terms of $dv$, $dr$, and $d\Omega^2$.

**g)** Kepler's third law is $\Omega^2 = M/r^3$, where $\Omega = \frac{d\phi}{dt}$. Find the corresponding result in the spacetime described by the line element (1).

**h)** Calculate the gravitational redshift of a photon emitted in the radial direction by a stationary observer at $r_A > M$ and received at $r = \infty$. The emitted photon has frequency $\omega_A$ and the received photon has frequency $\omega_\infty$.

## Problem 2

Consider the two-dimensional spacetime with the line element

$$ds^2 = -X^2 dT^2 + dX^2 \ . \tag{5}$$

**a)** Calculate the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$.

**b)** Calculate the diagonal components of the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar $R$.

**c)** Is the the spacetime with the line element (5) flat? Prove your claim!

## Problem 3

**a)** Write down the equation that defines parallel transport of a vector field $A^\alpha$ along a curve whose coordinates are $x^\alpha(\sigma)$, where $\sigma$ is a parameter. Use this equation to give a definition of geodesics in terms of parallel transport.

**b)** The covariant derivative of a covariant tensor of rank two $t_{\alpha\beta}$ is defined by

$$\nabla_\gamma t_{\alpha\beta} = \partial_\gamma t_{\alpha\beta} - \Gamma^\delta_{\alpha\gamma} t_{\beta\delta} - \Gamma^\delta_{\beta\gamma} t_{\alpha\delta} \ . \tag{6}$$

Calculate the covariant derivative of the metric tensor $g_{\alpha\beta}$.

## Problem 4

The line element for an isotropic and homogeneous universe is

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \left\{\begin{array}{l} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{array}\right\}(d\theta^2 + \sin^2\theta d\phi^2)\right] \ , \ \left\{\begin{array}{l} k=1 \\ k=0 \\ k=-1 \end{array}\right\} .$$

**a)** Explain briefly the terms isotropic and homogeneous. The spatial geometry of an isotropic and homogeneous universe is given by the value of $k$. Describe briefly the different geometries given by $k = 0, \pm 1$.

**b)** What is $a(t)$? What is the time dependence of $a(t)$ in a universe that has only a positive constant vacuum energy density $\Lambda$?

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Useful formulas

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left[ \frac{\partial g_{\alpha\delta}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\delta}} \right] , \tag{7}$$

$$R_{\alpha\beta} = \partial_{\gamma} \Gamma^{\gamma}_{\alpha\beta} - \partial_{\beta} \Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\alpha\beta} \Gamma^{\delta}_{\gamma\delta} - \Gamma^{\delta}_{\beta\gamma} \Gamma^{\gamma}_{\alpha\delta} . \tag{8}$$