Formalities.
Solutions should be handed in Wednesday 28.10., latest 14.00, in my mailbox (D5-166), by email or in the lectures.

The Reissner-Nordström Solution for a Charged Black Hole.
In this home exam, you will derive step-by-step the solution of the coupled Einstein-Maxwell equations for a point-like particle with mass M and electric charge Q.

a.) Show that a static, isotropic metric can be written as

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2)\,. \qquad (1)$$

b.) Show that the non-zero components of the Ricci tensor in this metric are given by

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \qquad (2)$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \qquad (3)$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \qquad (4)$$

$$R_{33} = R_{22}\sin^2\vartheta, \qquad (5)$$

where we order coordinates as $x^\mu = (t, r, \vartheta, \phi)$. You may use a program of your choice to do this calculation; if so, attach the code/output you used/produced. If you do the calculation "by hand", it is sufficient to calculate one of the 4 non-zero elements.

c.) Consider next the inhomogenous Maxwell equation for a point charge in the metric (1). Show that the inhomogeneous Maxwell equation,

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}F^{\mu\nu}) = j^\nu,$$

implies for the electric field

$$E(r) = \frac{\sqrt{AB}Q}{4\pi r^2}.$$

d.) Determine the non-zero contributions of the electric field to the stress tensor $T_{\mu\nu}$ and show that the Einstein equations simplify (using also $\Lambda = 0$) to $R_{\mu\nu} = -\kappa T_{\mu\nu}$. Give the explicit form of the the Einstein equations.

e.) Combine the R_{00} and R_{11} equations, and use (2) and (3) to show that $A(r)B(r) = 1$.

f.) Use the R_{22} component to determine $A(r)$ and $B(r)$.

g.) What are the physical and coordinate singularities, the horizons of the Reissner-Nordström BH solution?

Sign convention.
The signs of the metric tensor, Riemann's curvature tensor and the Einstein tensor can be fixed arbitrarily,

$$\eta^{\alpha\beta} = S_1 \times [-1, +1, +1, +1], \tag{6a}$$

$$R^{\alpha}{}_{\beta\rho\sigma} = S_2 \times [\partial_\rho \Gamma^{\alpha}{}_{\beta\sigma} - \partial_\sigma \Gamma^{\alpha}{}_{\beta\rho} + \Gamma^{\alpha}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\beta\sigma} - \Gamma^{\alpha}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\beta\rho}], \tag{6b}$$

$$S_3 \times G_{\alpha\beta} = 8\pi G\, T_{\alpha\beta}, \tag{6c}$$

$$R_{\alpha\beta} = S_2 S_3 \times R^{\rho}{}_{\alpha\rho\beta}. \tag{6d}$$

Here we choose these three signs as $S_i = \{-, +, -\}$.