NTNU – Trondheim
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# Solutions Exam FY3452 Gravitation and Cosmology summer 2018

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Permitted examination support material:
Rottmann: Matematisk Formelsamling
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Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

## Problem 1

**a)** The nonzero components of the metric can be read off from the line element and are

$$g_{tt} = \underline{\underline{-f(r)}}\,, \qquad g_{rr} = \underline{\underline{\frac{1}{f(r)}}}\,, \qquad g_{\phi\phi} = \underline{\underline{r^2}}\,. \tag{1}$$

The metric is diagonal. Since the metric is indenpendent of $t$ and $\phi$, there are (at least) two Killing vectors. These are

$$\boldsymbol{\xi} = \underline{\underline{(1,0,0)}}\,, \qquad \boldsymbol{\eta} = \underline{\underline{(0,0,1)}}\,. \tag{2}$$

The associated conserved quantities are $\boldsymbol{u}\cdot\boldsymbol{\xi}$ and $\boldsymbol{u}\cdot\boldsymbol{\eta}$

$$e = -\boldsymbol{u}\cdot\boldsymbol{\xi} = \underline{\underline{f(r)\frac{dt}{d\tau}}} \,, \qquad l = \boldsymbol{u}\cdot\boldsymbol{\eta} = \underline{\underline{r^2\frac{d\phi}{d\tau}}} \,. \tag{3}$$

Time independence implies energy conservation, while independence of $\phi$ implies conservation of the $z$-component of the angular momentum.

**b)** The Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$ can be calculated from the equation of motion

$$\frac{d}{d\sigma}\left[\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\sigma}\right)}\right] = \frac{\partial L}{\partial x^\mu} \,, \tag{4}$$

where $L = \left(-g_{\alpha\beta}\frac{dx}{d\sigma}^{\alpha}\frac{dx}{d\sigma}^{\beta}\right)^{\frac{1}{2}}$. We first consider $\mu = t$. Since $L$ is independent of $t$, the right-hand side of equation (4) vanishes. We find

$$\frac{\partial L}{\partial\left(\frac{dx^t}{d\sigma}\right)} = -\frac{1}{L} f(r)\frac{dx^t}{d\sigma} \tag{5}$$

Using $\frac{d\tau}{d\sigma} = L$, we can write $\frac{\partial L}{\partial\left(\frac{dx^t}{d\sigma}\right)} = f(r)\frac{dx^t}{d\tau}$ and the equation of motion becomes

$$L\frac{d}{d\tau}\left[f(r)\frac{dx^t}{d\tau}\right] = 0 \,. \tag{6}$$

This yields

$$f\frac{d^2x^t}{d\tau^2} + f'\frac{dx^t}{d\tau}\frac{dx^t}{d\tau} = 0 \,, \tag{7}$$

or

$$\frac{d^2x^t}{d\tau^2} + \frac{f'}{f}\frac{dx^t}{d\tau}\frac{dx^t}{d\tau} = 0 \,, \tag{8}$$

We can then read off the nonzero Christoffel symbols with $\gamma = t$

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \underline{\underline{\frac{1}{2}\frac{f'}{f}}} \,, \tag{9}$$

where the prime indicates differentiation with respect to $r$. The equations of motion for $\alpha = r$ and $\alpha = \phi$ can be calculated in the same manner. We list them for completeness

$$\frac{d^2r}{d\tau^2} + \frac{1}{2}ff'\frac{dt}{d\tau}\frac{dt}{d\tau} - \frac{1}{2}\frac{f'}{f}\frac{dr}{d\tau}\frac{dr}{d\tau} - rf\frac{d\phi}{d\tau}\frac{d\phi}{d\tau} = 0 \,, \tag{10}$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} = 0 \,. \tag{11}$$

This gives

$$\Gamma^r_{tt} = \underline{\underline{\frac{1}{2}ff'}} , \qquad \Gamma^r_{rr} = \underline{\underline{-\frac{1}{2}\frac{f'}{f}}} , \qquad \Gamma^r_{\phi\phi} = \underline{\underline{-rf}} \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \underline{\underline{\frac{1}{r}}} . \tag{12}$$

The other Christoffel symboles are zero.

**c)** For $\alpha = \beta = \phi$, we find

$$\begin{aligned} R_{\phi\phi} &= \partial_\gamma \Gamma^\gamma_{\phi\phi} - \partial_\phi \Gamma^\gamma_{\phi\gamma} + \Gamma^\gamma_{\phi\phi}\Gamma^\delta_{\gamma\delta} - \Gamma^\delta_{\phi\gamma}\Gamma^\gamma_{\phi\delta} \\ &= \partial_r \Gamma^r_{\phi\phi} + \Gamma^\gamma_{\phi\phi}\Gamma^\delta_{\gamma\delta} - \Gamma^\delta_{\phi\gamma}\Gamma^\gamma_{\phi\delta} \\ &= \partial_r \Gamma^r_{\phi\phi} + \Gamma^r_{\phi\phi}\left[\Gamma^r_{rr} + \Gamma^\phi_{r\phi} + \Gamma^t_{rt}\right] - \Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r} - \Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} \\ &= \partial_r\left[-rf\right] + \frac{1}{2}rf' + f - \frac{1}{2}rf' \\ &= \underline{\underline{-rf'}} . \end{aligned} \tag{13}$$

The other diagonal components of $R_{\alpha\beta}$ can be calculated in the same manner. This yields

$$R_{tt} = \underline{\underline{\frac{1}{2}ff'' + \frac{1}{2}\frac{ff'}{r}}} , \tag{14}$$

$$R_{rr} = \underline{\underline{-\frac{1}{2}\frac{f''}{f} - \frac{1}{2}\frac{f'}{fr}}} , \tag{15}$$

Contracting $R_{\alpha\beta}$ with the metric yields

$$R = \underline{\underline{-f'' - 2\frac{f'}{r}}} . \tag{16}$$

**d)** The Einstein equation in vacuum reads

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0 . \tag{17}$$

This yields

$$\frac{ff'}{r} = 0 , \tag{18}$$

$$\frac{f'}{fr} = 0 , \tag{19}$$

$$f''r^2 = 0 . \tag{20}$$

Thus $f$ is constant. We identify the line element as that of Minkowski spacetime for

$$f = \underline{\underline{1}} . \tag{21}$$

## Problem 2

**a)** We first calculate the differentials in the new coordinates

$$d\phi' = d\phi - \Omega dt \; . \tag{22}$$

We can therefore make the substitution $d\phi \to d\phi + \Omega dt$ in the metric. This yields

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2(d\phi - \Omega dt)^2 + dz^2 \\ &= -(1 - \Omega^2 r^2)dt^2 + dr^2 - 2\Omega r^2 d\phi dt + r^2 d\phi^2 + dz^2 \; . \end{aligned} \tag{23}$$

The relations $r = \sqrt{x^2 + y^2}$ and $\phi = \arctan \frac{y}{x}$ yield

$$dr = \frac{xdx}{\sqrt{x^2+y^2}} + \frac{ydy}{\sqrt{x^2+y^2}} \; , \tag{24}$$

$$d\phi = \frac{xdy - ydx}{x^2+y^2} \; . \tag{25}$$

Inserting Eqs. (24) and (25) into (23) and cleaning up, we obtain

$$ds^2 = \underline{\underline{-[1 - \Omega^2(x^2+y^2)]dt^2 + 2\Omega(ydx - xdy)dt + dx^2 + dy^2 + dz^2}} \; . \tag{26}$$

**b)** In the nonrelativistic limit, we can approximate $\tau = t$. This implies $\frac{dt}{d\tau} = 1$ and $\frac{d^2t}{d\tau^2} = 0$. This yields

$$\frac{d^2x}{dt^2} - 2\Omega\frac{dy}{dt} - \Omega^2 x = 0 \; , \tag{27}$$

$$\frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} - \Omega^2 y = 0 \; , \tag{28}$$

$$\frac{d^2z}{d^2t} = 0 \; . \tag{29}$$

**c)** Eq. (27) can be written as

$$\mathbf{a}_x - 2(\mathbf{\Omega} \times \mathbf{v})_x - (\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}))_x \; , \tag{30}$$

where the subscript $x$ means the $x$-component. Eqs. (28) and (29) can be written as the $y$ - and $z$-components of the same equation. Thus, we have

$$\mathbf{a} - 2\mathbf{A} \times \mathbf{v} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = 0 \; . \tag{31}$$

Being in a rotating frame of reference, fictitious forces are present. The term $-2\mathbf{\Omega} \times \mathbf{v}$ is the Coriolis force, while the term $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ is the centrifugal force.

## Problem 3

**a)** We know that the covariant derivative of a scalar is the the usual partial derivative. If $B_\beta$ is a covariant vector $s = A^\beta B_\beta$ is a scalar and we can write

$$\Delta_\alpha s = \frac{\partial s}{\partial x^\alpha} . \tag{32}$$

Using the Lebniz' rule we can also write

$$\begin{aligned} \Delta_\alpha A^\beta B_\beta &= \left(\nabla_\alpha A^\beta\right) B_\beta + A^\beta \left(\nabla_\alpha B_\beta\right) \\ &= \frac{\partial s}{\partial x^\alpha} \\ &= A^\beta \frac{\partial B_\beta}{\partial x^\alpha} + \frac{\partial A^\beta}{\partial x^\alpha} B_\beta . \end{aligned} \tag{33}$$

Substituting the expression for the covariant derivative a contravariant vector in Eq. (33), we find

$$A^\beta \left(\nabla_\alpha B_\beta\right) + \Gamma^\beta_{\alpha\gamma} A^\gamma B_\beta = A^\beta \frac{\partial B_\beta}{\partial x^\alpha} . \tag{34}$$

Swapping dummy indices $\beta$ and $\gamma$, this reads

$$A^\beta \left(\nabla_\alpha B_\beta\right) + \Gamma^\gamma_{\alpha\beta} A^\beta B_\gamma = A^\beta \frac{\partial B_\beta}{\partial x^\alpha} . \tag{35}$$

Since $A^\beta$ is arbitrary, we must have

$$\nabla_\alpha B_\beta = \underline{\underline{\frac{\partial B_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta} B_\gamma}} . \tag{36}$$

**b)** As a photon propagates in a gravitational field, its frequency $\omega$ changes. For example, if a photon propagates radially outwards in a Schwarzschild spacetime being emitted at $r_A$ and being detected at $r_B$, the frequencies are related as

$$\omega_B = \underline{\underline{\sqrt{\frac{1 - \frac{2M}{r_A}}{1 - \frac{2M}{r_B}}}}} , \tag{37}$$

where $M$ is the mass of the planet. Since $r_B > r_B$, we find $\omega_B < \omega_A$, i.e. gravitational redshift.

**c)** If an observer sees the same universe in all directions, it is isotropic around the point in space of the observer. If it is isotropic for all observers in the universe, it is globally isotropic.

If all observers see the same universe, it is homogeneous. These concepts are not equivalent. A uniform magnetic field in one direction, clearly breaks isotropy, but the universe can still be homogeneous.

**d)** The term $F_{\mu\nu}F^{\mu\mu}$ is gauge invariant as it is constructed out of the field tensor, which we know is invariant. The second term transforms as

$$\begin{aligned} j_\mu A^\mu &\rightarrow j_\mu A^{\mu\prime} \\ &= j_\mu \left(A^\mu + \partial^\mu \chi\right). \end{aligned} \tag{38}$$

where $\chi$ is a well-behaved function. The change is

$$\Delta\mathcal{L} = \underline{\underline{j_\mu \partial^\mu \chi}}\,. \tag{39}$$

The action also changes

$$\begin{aligned} \Delta S &= \int d^4x \Delta\mathcal{L} \\ &= \int d^4x j_\mu \partial^\mu \chi \end{aligned} \tag{40}$$

This can be written as

$$\begin{aligned} \Delta S &= \int d^4x \left[\partial^\mu \left(\chi j_\mu\right) - \chi \partial^\mu j_\mu\right] \\ &= \int d^4x \left[\partial^\mu \left(\chi j_\mu\right)\right] \,, \end{aligned} \tag{41}$$

where we have used current conservation, $\partial^\mu j_\mu = 0$. The Lagrangian changes by a total derivative, which is allowed. The action does not change.

## Problem 4

We denote the ejected four-momentum by $\mathbf{p}_e$ and the remaining four-momentum by $\mathbf{p}_f$. The initial four-momentum is denoted by $\mathbf{p}$. Conservation of four-momentum gives

$$\mathbf{p} = \mathbf{p}_e + \mathbf{p}_f\,. \tag{42}$$

This yields

$$p_e^2 = -m^2 - m_f^2 - 2\mathbf{p}_f \cdot \mathbf{p} \tag{43}$$

Since the ejected material has zero rest mass, we have $p_e^2 = 0$. The initial four-momentum $\mathbf{p}$ is (spaceship at rest)

$$\mathbf{p} = m\left(\frac{1}{\sqrt{1-\frac{2M}{r}}}, 0, 0, 0\right)\,. \tag{44}$$

To evaluate the product $\mathbf{p}_f \cdot \mathbf{p}$, we only need the zeroth component of the four momentum $\mathbf{p}_f$. This denoted by $p^t_f(r)$. Conservation of energy gives

$$p^t_f(r)\left(1 - \frac{2M}{r}\right) \quad = \quad m_f e \; . \tag{45}$$

The spaceship must be at rest at $r = \infty$, whence $e = 1$.

$$p^t_f(r) \quad = \quad \frac{m_f}{1 - \frac{2M}{r}} \; . \tag{46}$$

Writing $m_f = mf$, where $f$ is the fraction, and using the expressions for the four-momentum $\mathbf{p}$ , Eq. (43) can be written as

$$m^2(1 + f^2) - \frac{2m^2 f}{\sqrt{1 - \frac{2M}{r}}} \quad = \quad 0 \; . \tag{47}$$

The solution for $f$ is

$$f \quad = \quad \frac{1 \pm \sqrt{\frac{2M}{R}}}{\sqrt{1 - \frac{2M}{R}}} \; . \tag{48}$$

The positive solution yields $f > 1$, which must be rejected on physical grounds. Hence, the fraction is

$$f \quad = \quad \underline{\underline{\frac{1 - \sqrt{\frac{2M}{R}}}{\sqrt{1 - \frac{2M}{R}}}}} \; . \tag{49}$$

The limit is

$$\begin{aligned} f_{\text{horizon}} \quad &= \quad \lim_{R \to 2M} f \\ &= \quad \underline{\underline{0}} \; . \end{aligned} \tag{50}$$

Thus, nothing can escape if the spaceship starts at the horizon.