NTNU – Trondheim
Norwegian University of
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Solutions Exam FY3452 Gravitation and Cosmology fall 2017

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Permitted examination support material:
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

Problem 1

a) The singularities are $r = 0$ and $r = M$. In analogy with the Schwarzschild we expect $r = 0$ to be a physical singularity and $R = M$ to be a coordinate singularity. No proof is required, but the latter is shown in **e**).

b) Since the metric is indenpendent of t and ϕ, there are (at least) two Killing vectors. These are

$$\boldsymbol{\xi} = \underline{\underline{(1,0,0,0)}}\,, \qquad \boldsymbol{\eta} = \underline{\underline{(0,0,0,1)}}\,. \tag{1}$$

The associated conserved quantities are $\boldsymbol{u}\cdot\boldsymbol{\xi}$ and $\boldsymbol{u}\cdot\boldsymbol{\eta}$

$$e = -\boldsymbol{u}\cdot\boldsymbol{\xi} = \underline{\underline{\left(1-\frac{M}{r}\right)^2 \frac{dt}{d\tau}}}\,, \qquad l = \boldsymbol{u}\cdot\boldsymbol{\eta} = \underline{\underline{r^2\sin^2\theta\frac{d\phi}{d\tau}}}\,. \tag{2}$$

Time independence implies energy conservation, while independence of ϕ implies conservation of the z-component of the angular momentum. Hence, e and l are energy and angular momentum per unit mass, respectively.

c) The motion is confined to a plane and the coordinate system is chosen such that $\theta = \frac{\pi}{2}$. We first write use the normalization of the four-velocity of the particle as

$$\begin{aligned} -1 &= \mathbf{u}\cdot\mathbf{u} \\ &= -\left(1-\frac{M}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{M}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 . \end{aligned} \tag{3}$$

Eliminating $\frac{dt}{d\tau}$ and $\frac{d\phi}{d\tau}$ in favor of e and l, we can write

$$-\left(1-\frac{M}{r}\right)^{-2} e^2 + \left(1-\frac{M}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2 + \frac{r^2}{l^2} = -1\,. \tag{4}$$

This can be rewritten as

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r)\,, \tag{5}$$

where

$$V_{\text{eff}}(r) = \underline{\underline{\frac{1}{2}\left[\left(1-\frac{M}{r}\right)^2\left(\frac{l^2}{r^2}+1\right)-1\right]}}\,. \tag{6}$$

d) A particle starting at rest at $r=\infty$ has $e=1$. Since it is falling radially inwards, it has $l=0$. The minimum radius is obtained when $\frac{dr}{d\tau}=0$ and so r_{min} satisfies the equation $V_{\text{eff}}(r_{\text{min}})=0$. This yields

$$\left[\left(1-\frac{M}{r_{\text{min}}}\right)^2 - 1\right] = 0\,, \tag{7}$$

whose solution is

$$r_{\text{min}} = \underline{\underline{\frac{1}{2}M}}\,. \tag{8}$$

For a particle with $e=1$, we find

$$\frac{dr}{d\tau} = -\sqrt{1-\left(1-\frac{M}{r}\right)^2}\,, \tag{9}$$

where we chosen the minus sign since the particle is moving inwards. This yields

$$\begin{aligned}\Delta\tau &= -\int_M^{\frac{1}{2}M} \frac{dr}{\sqrt{1-\left(1-\frac{M}{r}\right)^2}} \\ &= \underline{\underline{\frac{2}{3}m}}\ . \end{aligned} \tag{10}$$

e) The line element can be written as

$$\begin{aligned} ds^2 &= -\left(1-\frac{M}{r}\right)^2\left[dt^2 - \frac{dr^2}{\left(1-\frac{M}{r}\right)^4}\right] + r^2 d\Omega^2 \\ &= -\left(1-\frac{M}{r}\right)^2\left[dt + \frac{dr}{\left(1-\frac{M}{r}\right)^2}\right]\left[dt - \frac{dr}{\left(1-\frac{M}{r}\right)^2}\right] + r^2 d\Omega^2 \\ &= -\left(1-\frac{M}{r}\right)^2\left[d\tilde{t} + dr\right]\left[d\tilde{t} + dr - \frac{2dr}{\left(1-\frac{M}{r}\right)^2}\right] + r^2 d\Omega^2\ . \end{aligned} \tag{11}$$

The radial light rays satisfy $ds^2 = 0$, which yields

$$d\tilde{t} + dr = 0\ , \tag{12}$$

$$d\tilde{t} + dr - \frac{2dr}{\left(1-\frac{M}{r}\right)^2} = 0\ , \tag{13}$$

The first equation gives $\tilde{t} + r =$ constant, which corresponds to incoming light. The second equation gives

$$\frac{d\tilde{t}}{dr} = \frac{2}{\left(1-\frac{M}{r}\right)^2} - 1\ . \tag{14}$$

This is an outgoing curve for $r > M$. It also an outgoing curve for $M > r > \frac{M}{\sqrt{2}+1}$, but it never crosses $r = M$ since $\frac{d\tilde{t}}{dr}$ diverges as $r \to M^{-1}$. Light can therefore cross from the region $r > M$ to the region $r < M$ but not from $r < M$ to $r > M$. *The line element therefore describes the geometry of a black hole.* These curves are sketched in Fig. 1.

f) Substituting $d\tilde{t} = dv - dr$, the line element now becomes

$$ds^2 = \underline{\underline{-\left(1-\frac{M}{r}\right)^2 dt^2 + 2dvdr + r^2 d\Omega^2}}\ . \tag{15}$$

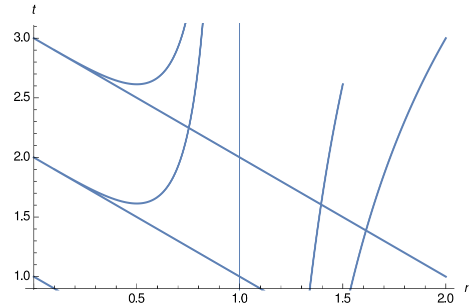

Figure 1: $(\tilde{t}, r)$ diagram.

g) We need to calculate

$$\begin{aligned}\Omega &= \frac{d\phi}{dt} \\ &= = \frac{d\phi}{d\tau}\frac{d\tau}{dt} \\ &= \frac{l}{r^2}\frac{(1-\frac{M}{r})^2}{e} \, . \end{aligned} \tag{16}$$

A *stable* circular orbit with radius r has $\frac{dr}{d\tau} = 0$ where r is a minimum of the effective potential. It therefore satisfies

$$\frac{e^2-1}{2} = V_{\text{eff}}(r) \, , \tag{17}$$

$$V'_{\text{eff}}(r) = 0 \, . \tag{18}$$

This yields

$$\frac{l^2}{e^2} = \frac{Mr}{\left(1-\frac{M}{r}\right)^3} \, . \tag{19}$$

Inserting Eq. (19) into Eq. (16), we find

$$\Omega^2 = \underline{\underline{\frac{M}{r^3}\left(1-\frac{M}{r}\right)}} \, . \tag{20}$$

In contrast to the Schwarzschild spacetime, this result is not of the same form as Kepler's third law.

h) The four-velocity of the stationary observer is

$$\begin{aligned}\mathbf{u}_{\text{obs}} &= \left(\frac{1}{1-\frac{M}{r}}, 0, 0, 0\right) \\ &= \frac{1}{1-\frac{M}{r}}\xi \, . \end{aligned} \tag{21}$$

The energy of the photon is $E = \hbar\omega = -\mathbf{p}\cdot\mathbf{u}_{\text{obs}}$, where $\mathbf{p}$ is the four-momentum of the photon. Since $\xi\cdot\mathbf{p}$ is constant along the photon's trajectory, we find

$$\begin{aligned}\hbar\omega\left(1-\frac{M}{r}\right) &= \xi\cdot\mathbf{p} \\ &= \text{constant}\ . \end{aligned} \tag{22}$$

This yields

$$\omega_\infty = \underline{\underline{\omega_A\left(1-\frac{M}{r}\right)}}\ . \tag{23}$$

In the limit $r_A \to M$, the redshift is infinite.

Problem 2

a) The Lagrangian for the geodesic is given by

$$L = \sqrt{-X^2\left(\frac{dT}{d\sigma}\right)^2 + \left(\frac{dX}{d\sigma}\right)^2}\ . \tag{24}$$

Using the Euler-Lagrange equations and the fact that $L = \frac{d\tau}{d\sigma}$, we get the geodesic equations

$$\frac{d}{d\tau}\left(X^2\frac{dT}{d\tau}\right) = 0 \quad , \tag{25}$$

$$\frac{d^2X}{d\tau^2} + X\left(\frac{dT}{d\tau}\right)^2 = 0 \quad . \tag{26}$$

We can then read off the nonzero Christoffel symbols

$$\Gamma^T_{TX} = \Gamma^T_{XT} = \underline{\underline{\frac{1}{X}}}\ , \tag{27}$$

$$\Gamma^X_{TT} = \underline{\underline{X}}\ . \tag{28}$$

b) We first consider R_{TT}, which equals

$$\begin{aligned} R_{TT} &= \partial_\gamma\Gamma^\gamma_{TT} - \partial_T\Gamma^\gamma_{T\gamma} + \Gamma^\gamma_{TT}\Gamma^\delta_{\gamma\delta} - \Gamma^\delta_{T\gamma}\Gamma^\gamma_{T\delta} \\ &= \partial_X\Gamma^X_{TT} + \Gamma^X_{TT}\Gamma^\delta_{X\delta} - \Gamma^\delta_{TT}\Gamma^T_{T\delta} - \Gamma^\delta_{TX}\Gamma^X_{T\delta} \\ &= \partial_X X + X\frac{1}{X} - X\frac{1}{X} - X\frac{1}{X} \\ &= \underline{\underline{0}}\ . \end{aligned} \tag{29}$$

We can calculate the R_{XX} in the same way and find $R_{XX} = \underline{\underline{0}}$. This trivially yields $R = \underline{\underline{0}}$.

c) Yes, the line element describes Minkowski space. By introducing the coordinates x and t via

$$t = X \sinh T \,, \tag{30}$$

$$x = X \cosh T \,, \tag{31}$$

the line element becomes

$$ds^2 = -dt^2 + dx^2 \,. \tag{32}$$

Problem 3

a) The vector field A^α satisfies the equation

$$\frac{dA^\alpha}{d\sigma} + \Gamma^\alpha_{\beta\gamma} A^\beta \frac{dx^\gamma}{d\sigma} = 0 \,, \tag{33}$$

where $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol and $\frac{dx^\gamma}{d\sigma}$ is the γ-component of the tangent vector to the curve parametrized by the parameter σ. Set $A^\beta = \frac{dx^\beta}{d\sigma}$ and we find

$$\frac{d^2 x^\alpha}{d\sigma^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\sigma} \frac{dx^\gamma}{d\sigma} = 0 \,, \tag{34}$$

Thus a geodesic is a curve, whose tangent vector is being parallel transported along the curve.

b) The second term in the covariant derivative is

$$\begin{aligned}
\Gamma^\delta_{\alpha\gamma} g_{\beta\delta} &= \frac{1}{2} g^{\delta\rho} \left[\frac{\partial g_{\alpha\rho}}{\partial x^\gamma} + \frac{\partial g_{\gamma\rho}}{\partial x^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x^\rho} \right] g_{\beta\delta} \\
&= \frac{1}{2} \delta^\rho_\beta \left[\frac{\partial g_{\alpha\rho}}{\partial x^\gamma} + \frac{\partial g_{\gamma\rho}}{\partial x^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x^\rho} \right] \\
&= \frac{1}{2} \left[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} \right] \,.
\end{aligned} \tag{35}$$

The third term can be found by swapping α and β,

$$\Gamma^\delta_{\beta\gamma} g_{\alpha\delta} = \frac{1}{2} \left[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right] \,. \tag{36}$$

Adding the first term $\partial_\gamma g_{\alpha\beta}$ to Eqs. (35)–(36), we find

$$\nabla_\gamma g_{\alpha\beta} = \underline{\underline{0}} \,. \tag{37}$$

The metric tensor is covariant constant.

Problem 4

a) **Isotropic** means that the universe looks the same in all directions from a given point in space, while **homogeneous** means that the the universe looks the same from every point in the universe. If the universe is globally isotropic, it is isotropic around every point. These concepts are *not* equivalent. A constant magnetic field breaks isotropy, but the universe can never the be homogeneous.

$k = 0$ corresponds to flat three-dimensional Euclidean space. $k = 1$ corresponds to the geometry of a 3-sphere embedded in a four-dimensional Euclidean space. $k = -1$ corresponds to a three-dimensional hyperboloid embedded in flat four-dimensional Minkowski space.

b) $a(t)$ is the socalled scale factor. Once $a(t)$ is determined, the dynamics of the homogeneous and isotropic universe models are completely determined. The scale factor in a universe wiht only constant positve vacuum energy is an exponential, $a(t) \sim e^{\sqrt{\Lambda} t}$. The universe is expanding exponentially, which is referred to as *inflation*.