Formalities.
Solutions should be handed in Wednesday 28.10., latest 14.00, in my mailbox (D5-166), by email or in the lectures.

The Reissner-Nordström Solution for a Charged Black Hole.
In this home exam, you will derive step-by-step the solution of the coupled Einstein-Maxwell equations for a point-like particle with mass M and electric charge Q.

a.) Show that a static, isotropic metric can be written as

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2)\,. \tag{1}$$

b.) Show that the non-zero components of the Ricci tensor in this metric are given by

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \tag{2}$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \tag{3}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \tag{4}$$

$$R_{33} = R_{22}\sin^2\vartheta, \tag{5}$$

where we order coordinates as $x^\mu = (t, r, \vartheta, \phi)$. You may use a program of your choice to do this calculation; if so, attach the code/output you used/produced. If you do the calculation "by hand", it is sufficient to calculate one of the 4 non-zero elements.

c.) Consider next the inhomogenous Maxwell equation for a point charge in the metric (1). Show that the inhomogeneous Maxwell equation,

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}F^{\mu\nu}) = j^\nu,$$

implies for the electric field

$$E(r) = \frac{\sqrt{AB}Q}{4\pi r^2}.$$

d.) Determine the non-zero contributions of the electric field to the stress tensor $T_{\mu\nu}$ and show that the Einstein equations simplify (using also $\Lambda = 0$) to $R_{\mu\nu} = -\kappa T_{\mu\nu}$. Give the explicit form of the the Einstein equations.

e.) Combine the R_{00} and R_{11} equations, and use (2) and (3) to show that $A(r)B(r) = 1$.

f.) Use the R_{22} component to determine $A(r)$ and $B(r)$.

g.) What are the physical and coordinate singularities, the horizons of the Reissner-Nordström BH solution?

a.) A stationary spacetime has a time-like Killing vector field. In appropriate coordinates, the metric tensor is therefore independent of the time coordinate,

$$\mathrm{d}s^2 = g_{00}(\boldsymbol{x})\mathrm{d}t^2 + 2g_{0i}(\boldsymbol{x})\mathrm{d}t\mathrm{d}x^i + g_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j\,. \tag{6}$$

A stationary spacetime is static if it is invariant under time reversal. Thus the off-diagonal terms g_{0i} have to vanish, and the metric simplifies to

$$\mathrm{d}s^2 = g_{00}(\boldsymbol{x})\mathrm{d}t^2 + g_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j. \tag{7}$$

Finally, isotropy requires that only $r^2 \equiv \boldsymbol{x}\cdot\boldsymbol{x}$, $\mathbf{d}\boldsymbol{x}\cdot\boldsymbol{x}$, and $\mathbf{d}\boldsymbol{x}\cdot\mathbf{d}\boldsymbol{x}$ enter the spatial line-element $\mathrm{d}l^2$. Hence

$$\mathrm{d}l^2 = C(r)(\boldsymbol{x}\cdot\mathbf{d}\boldsymbol{x})^2 + D(r)(\mathbf{d}\boldsymbol{x}\cdot\mathbf{d}\boldsymbol{x})^2 = C(r)r^2\mathrm{d}r^2 + D(r)[\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\phi^2] \tag{8}$$

We can eliminate the function $D(r)$ by the rescaling $r^2 \to Dr^2$. Thus the line-element becomes with $g_{00}(r) = A(r)$

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2)\,. \tag{9}$$

c.) The electric field measured by an observer with four-velocity u_α is $E_\alpha = F_{\alpha\beta}u^\beta$, i.e. it is associated with the field-strength tesnor with lowe indices. For a static charge at $r = 0$, the only non-zero field component is the radial elecric field, $F_{01} = -F_{10} = E(r)$. Raising the indices, we have $F^{01} = g^{00}g^{11}F^{01} = E/(AB)$. Next we determine the determinant of the metric, obtaining

$$\sqrt{|g|} = \sqrt{AB}r^2\sin^2\vartheta. \tag{10}$$

For $r > 0$, the current j^μ is zero and thus

$$\partial_r\left(\sqrt{AB}r^2F^{01}\right) = \partial_r\left(\frac{r^2E}{\sqrt{AB}}\right) = 0 \tag{11}$$

can be integrated, with the result

$$E(r) = \frac{\sqrt{AB}Q}{4\pi r^2}. \tag{12}$$

Here, the integration constant k can be identified with $Q/(4\pi)$, because of $A, B \to 1$ for $r \to \infty$. [The homogenous Maxwell equation is automatically satisfied, since there exists a potential A_0 with $E(r) = -\partial_r A_0$.]

d.) For a free electromagnetic field, it is always

$$T_\mu^{\ \mu} = -F_{\mu\sigma}\,F^{\mu\sigma} + \frac{1}{4}\,\delta^{\mu\mu}F_{\sigma\tau}F^{\sigma\tau} = 0\,. \tag{13}$$

Thus the Einstein equation becomes

$$R_{\mu\nu} = -\kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu}\,. \tag{14}$$

Evaluating

$$T_{\mu\nu} = -F_{\mu\sigma}\,F_\nu^{\ \sigma} + \frac{1}{4}\,\eta_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau} \tag{15}$$

using also $F_0{}^1 = g^{11}F_{01} = E/B$ and $F_1{}^0 = g^{00}F_{10} = E/A$, we find that they are given explicitely (setting $\kappa = 1$) by

$$R_{00} = -E^2/(2B), \tag{16}$$
$$R_{11} = E^2/(2A), \tag{17}$$
$$R_{22} = -E^2r^2/(2AB), \tag{18}$$
$$R_{33} = R_{22}\sin^2\vartheta. \tag{19}$$

e.) Combining (16) and (17), it follows

$$BR_{00} + AR_{11} = 0. \tag{20}$$

Next we insert (2) and (3), obtaining

$$A'B + AB' = 0. \tag{21}$$

Then $(AB)' = 0$ or $AB = \text{const}$. Assuming that spacetime becomes Minkowskian at large distance, it follows $A(r)B(r) = 1$.
Remark: We see now that the choice of the radial coordinate in (1) is such that r corresponds to the "luminosity distance" or, in other words, such that a $1/r^2$ law is valid for the flux from a point source.

f.) We insert $AB = 1$ and (12) into (18),

$$R_{22} = -E^2r^2/(2AB) = -\frac{1}{2}E^2r^2 = -\frac{1}{2}\frac{Q^2}{(4\pi)^2r^2}. \tag{22}$$

Next we simplify (4) using $AB = 1$, obtaining

$$R_{22} = A - 1 + A'r \tag{23}$$

and thus

$$A + A'r = 1 - \frac{1}{2}\frac{Q^2}{(4\pi)^2r^2}. \tag{24}$$

Using $A + A'r = (Ar)'$ and integrating, it follows

$$A(r) = 1 + \frac{k}{r} + \frac{Q^2}{2(4\pi)^2r^2} = 1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2} \quad \text{and} \quad B(r) = 1/A(r), \tag{25}$$

where the integration constant k was fixed by requiring that we obtain the Schwarzschild metric for $Q = 0$. In the last step, we changed also from $\kappa = 1$ to $\kappa = 8\pi G$.

g.) We set now $G = 1$. For $Q^2/(4\pi) < M^2$, two horizons are given by the solution of $A = 0$, or

$$r_\pm = M \pm \sqrt{M^2 - Q^2/(4\pi)}. \tag{26}$$

Possibly singularities are given by $B = 0$, i.e. at $r = 0$ and $r =$. Calculating the invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, we see that the latter is a coordinate singularity, while the former is a physical one.

Sign convention.
The signs of the metric tensor, Riemann's curvature tensor and the Einstein tensor can be fixed arbitrarily,

$$\eta^{\alpha\beta} = S_1 \times [-1, +1, +1, +1], \tag{27a}$$

$$R^{\alpha}{}_{\beta\rho\sigma} = S_2 \times [\partial_\rho \Gamma^{\alpha}{}_{\beta\sigma} - \partial_\sigma \Gamma^{\alpha}{}_{\beta\rho} + \Gamma^{\alpha}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\beta\sigma} - \Gamma^{\alpha}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\beta\rho}], \tag{27b}$$

$$S_3 \times G_{\alpha\beta} = 8\pi G\, T_{\alpha\beta}, \tag{27c}$$

$$R_{\alpha\beta} = S_2 S_3 \times R^{\rho}{}_{\alpha\rho\beta}. \tag{27d}$$

Here we choose these three signs as $S_i = \{-, +, -\}$.