NTNU Trondheim, Institutt for fysikk

Examination for FY3464 Quantum Field Theory I
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Allowed tools: Pocket calculator, mathematical tables
Some formulas can be found at the end of p.2.

1. Compton scattering in scalar QED.
Consider Compton scattering $\phi(p)+\gamma(k)\to\phi(p')+\gamma(k')$ between a scalar particle ϕ with mass m and a photon.
a. Draw all Feynman diagrams and write down the S-matrix element S_{fi} of this process at lowest order perturbation theory. Show that there is a gauge, where the squared Feynman amplitude $|\mathcal{M}|^2$ simplifies to

$$|\mathcal{M}|^2 = 4e^4(\varepsilon'\cdot\varepsilon)^2\,.$$

b. Give *one* reason why there exists a vertex of type c (including 2 photons) in scalar QED, but not in QED with fermions.
c. Estimate the size of total cross section σ for this process in cm^2 for $m = 100\,\mathrm{GeV}$.

a.

$$\begin{aligned} S_{fi} &= (-\mathrm{i}e)^2(2\pi)^4\delta^{(4)}(p+k-p'-k')\Big[\varepsilon'\cdot(2p'+k)\,\frac{\mathrm{i}}{(p+k)^2-m^2}\,\varepsilon\cdot(2p+k) \\ &\quad +\varepsilon\cdot(2p'-k)\,\frac{\mathrm{i}}{(p-k')^2-m^2}\,\varepsilon'\cdot(2p-k')-2\mathrm{i}\varepsilon'\cdot\varepsilon\Big]\,. \end{aligned} \tag{1}$$

Use transverse polarized photon in the rest frame of initial ϕ. Then $\varepsilon\cdot p=\varepsilon'\cdot p=0$. Since also $\varepsilon\cdot k=\varepsilon'\cdot k'=0$, the second and the fouth scalar products are obviously zero. Thus only the $\varepsilon'\cdot\varepsilon$ term survives,

$$\mathcal{M}^2 = 4e^4(\varepsilon'\cdot\varepsilon)^2\,. \tag{2}$$

b. Gauge interactions are derived replacing $\partial_\mu\to D_\mu=\partial_\mu-\mathrm{i}eA_\mu$ in the $\mathcal{L}_0$. This term is linear in the derivatives for Dirac particles, but quadratic for scalars. Thus the replacement in $\mathcal{L}_0=\partial_\mu\phi^\dagger\partial^\mu\phi$ generates an $e^2\phi^\dagger\phi A^\mu A_\mu$ term, in contrast to $\mathcal{L}_0=\mathrm{i}\bar\psi\gamma^\mu\partial_\mu\psi$ for Dirac fermions.
OR
Renormalizability allows only $d\le 4$ terms. With $[A]=[\phi]=m$ for bosonic fields, a four boson vertex has dimension 4 and is allowed, while $[\psi]=m^{3/2}$ leads to a dimension 5 operator.
OR
Direct calculation shows that without c) the matrix element is not invariant under gauge transformations, $\varepsilon^\mu\to\varepsilon^\mu+\lambda k^\mu$.

c. By dimensional reasons $\sigma \sim \alpha^2/\text{energy}^2$ and by Lorentz invariance $\sigma = \sigma(s, m^2)$. Thus $\sigma \sim \sigma_0 = \alpha^2/m^2$ at low energies $s \ll m^2$ and $\sigma \sim \sigma_0(m^2/s)$ for $s \gg m^2$.
Comment: There are cases where a cross section of the type $\sigma = \sigma(s, m^2, M^2)$ behaves as $\sigma \propto 1/M^2$ in the high-energy limit. A decision between the two cases is not possible without a more detailed discussion, and thus both $\sigma \sim \sigma_0$ and $\sigma \sim \alpha^2/s$ as answer for the high-energy behavior are considered as correct answers.

2. Radiative corrections in scalar QED.
a. Draw all Feynman diagrams of the "primitive divergent" graphs, i.e. the loop diagrams in lowest order perturbation theory.
b. Find the superficial degree of divergence D of the diagrams by power counting of loop momenta.

a and b. Scalar QED has two coupling constants. "Loop diagrams in lowest order perturbation theory" means all diagrams containing one loop. The full number of points was already given for a "representative" subset of diagrams. See page 6 for diagrams. The derivative scalar-photon coupling has to be included in the power-counting.
We order diagrams arrcording the number of external lines: i) Two external lines: $D = 2$ scalar self-energy and $D = 2$ photon vacuum polarization.
ii) Three external lines: photon-scalar-scaler vertex correction $D = 1$.
iii) Four external lines: scalar-scalar-scalar-scalar vertex correction $D = 0$.
Comment: One and three external photon lines vanish ("Furry theorem"); light-by-light scattering is finite; zero external lines ($D = 4$) correspond to contribution of the scalar and the photon to the cosmological constant.

3. Neutrino-electron scattering in the Fermi theory.
Consider neutrino-electron scattering $\bar{\nu}_e(k) + e^-(p) \to \bar{\nu}_e(k') + e^-(p')$ in the Fermi theory with the interaction

$$\mathcal{L}_I = \frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu$$

and

$$J^\mu = \bar{u}_e(p', s'_e)\gamma^\mu(1-\gamma^5)v_\nu(k', s'_\nu)$$

a. Write down the S-matrix element S_{fi} and the Feynman amplitude $\mathcal{M}$ of this process (neglecting m_ν in the nominator). (4 pts)
a. Sum/average the squared matrix element $|\mathcal{M}|^2$ over spins and show that it can be written as

$$\left|\overline{\mathcal{M}}\right|^2 = \frac{G_F}{2} M^{\mu\bar{\mu}}(p', k') N_{\mu\bar{\mu}}(p, k)$$

with

$$M^{\mu\bar{\mu}}(p', k') = 2k'_\alpha p'_\beta \,\text{tr}\{(1-\gamma^5)\gamma^\alpha\gamma^{\bar{\mu}}\gamma^\beta\gamma^\mu\}$$

a. With $\bar{u}(p') \equiv \bar{u}_e(p', s'_e)$, etc, it is

$$S_{fi} = -\mathrm{i}\frac{G_F}{\sqrt{2}}(2\pi)^4\delta^{(4)}(p+k-p'-k')\left[\bar{u}(p')\gamma^\mu(1-\gamma^5)v(k')\right]\left[\bar{v}(k)\gamma_\mu(1-\gamma^5)u(p)\right] \tag{3}$$

and

$$\mathcal{M}_{fi} = \frac{G_F}{\sqrt{2}}\left[\bar{u}(p')\gamma^\mu(1-\gamma^5)v(k')\right]\left[\bar{v}(k)\gamma_\mu(1-\gamma^5)u(p)\right] \tag{4}$$

b.

$$\begin{aligned}|\overline{\mathcal{M}}|^2 &= \frac{1}{2}\sum_{s_e,s_\nu,s'_e,s'_\nu}|\mathcal{M}|^2 = \frac{G_F^2}{4}\sum_{s_e,s_\nu}\left[\bar{u}(p')\gamma^\mu(1-\gamma^5)v(k')\bar{v}(k')\gamma^\mu(1+\gamma^5)u(p')\right] \\ &\times\left[\bar{v}(k))\gamma_\mu(1-\gamma^5)u(p)\bar{u}(p)\gamma_\mu(1+\gamma^5)v(k)\right] = \frac{G_F}{4}M^{\mu\bar{\mu}}(p',k')N_{\mu\bar{\mu}}(p,k)\end{aligned} \tag{5}$$

Thus

$$M^{\mu\bar{\mu}}(p',k') = \mathrm{tr}\{\not{k}'\gamma^\mu(1+\gamma^5)(\not{p}'+m_e)\gamma^\mu(1-\gamma^5)\} \tag{6}$$

With $\not{k}'(1+\gamma^5) = (1-\gamma^5)\not{k}'$ and $(1-\gamma^5)^2 = 2(1-\gamma^5)$ it follows

$$M^{\mu\bar{\mu}}(p',k') = 2\mathrm{tr}\{(1-\gamma^5)\not{k}'\gamma^\mu(\not{p}'+m_e)\gamma^\mu\} \tag{7}$$

The trace over an odd number of gamma matrices vnishes, and thus m_e does not contribute,

$$M^{\mu\bar{\mu}}(p',k') = 2\mathrm{tr}\{(1-\gamma^5)\not{k}'\gamma^\mu\not{p}'\gamma^\mu\} = 2k'_\alpha p'_\beta\,\mathrm{tr}\{(1-\gamma^5)\gamma^\alpha\gamma^{\bar{\mu}}\gamma^\beta\gamma^\mu\} \tag{8}$$

4. Non-abelian gauge transformation.

A gauge field $A_\mu = A^a_\mu T^a$ transforms as

$$A_\mu(x) \to A'_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \frac{\mathrm{i}}{g}U(x)\partial_\mu U^\dagger(x)$$

under a local gauge transformation $U(x) = \exp[-\mathrm{i}g\vartheta^a(x)T^a]$, where T^a are the generators of a Lie group with $[T^a, T^b] = \mathrm{i}f^{abc}T^c$.

Show that an infinitesimal gauge transformation,

$$U(x) = \exp(-\mathrm{i}g\vartheta^a(x)T^a) \to 1 - \mathrm{i}g\vartheta^a T^a + \mathcal{O}(\vartheta^2)$$

of the gauge field can be written as

$$A^a_\mu(x) \to A^{a\prime}_\mu(x) = A^a_\mu(x) - D^{ac}_\mu\vartheta^c(x)$$

with $D^{ac}_\mu \equiv \delta^{ac}\partial_\mu + gf^{abc}A^b_\mu(x)$.

For an infinitesimal gauge transformation,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \mathrm{i}g[A_\mu(x), \vartheta(x)] - \partial_\mu\vartheta(x) \tag{9}$$

Thus

$$
\begin{aligned}
A_\mu^a(x) \to A_\mu^{a\prime}(x) &= A_\mu^a(x) - g f^{abc} A_\mu^b(x) \vartheta^c(x) - \partial_\mu \vartheta^a(x) \\
&= A_\mu^a(x) - [\delta^{ac}\partial_\mu + g f^{abc} A_\mu^b(x)] \vartheta^c(x) \\
&\equiv A_\mu^a(x) - D_\mu^{ac} \vartheta^c(x) \,.
\end{aligned} \tag{10}
$$

$$\hbar c = 197.3 \text{MeV fm}$$

$$(\hbar c)^2 = 0.389 \text{GeV}^2 \text{ mbarn}$$

$$1 \text{mbarn} = 10^{-28} \text{m}^2$$

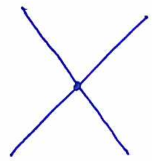
$$\sum_s u_a(p,s) \bar{u}_b(p,s) = \left(\frac{\not{p} + m}{2m} \right)_{ab}$$

$$\sum_s v_a(p,s) \bar{v}_b(p,s) = \left(\frac{\not{p} - m}{2m} \right)_{ab} \,.$$

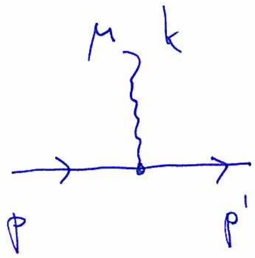
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$$

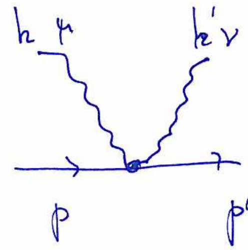
$$\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$$



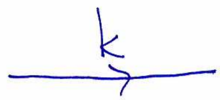
$-i\lambda$



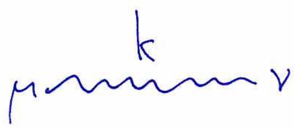
$-ie(p_\mu + p'_\mu)$



$2ie^2 g_{\mu\nu}$



$$\frac{i}{k^2-m^2+i\varepsilon}$$

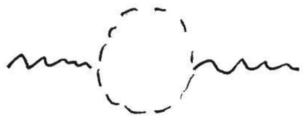


$$\frac{-i\,g_{\mu\nu}}{k^2+i\varepsilon}$$ (Feynman gauge)

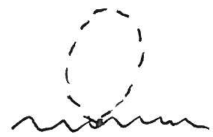
[N.B.: scalar particles solid line for better visibility]

$E=2$

photon self-energy

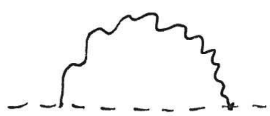


$D = 4+2-2\times 2 = 2$

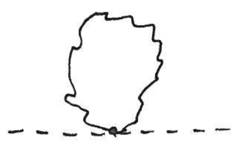


$D = 4-2 = 2$

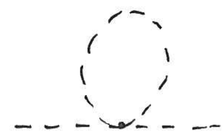
scalar self-energy:



$D = 4+2-2\times 2 = 2$

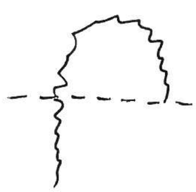


$4-2$

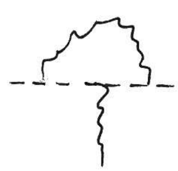


$4-2$

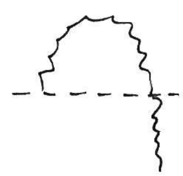
$E=3$ $\gamma\phi\phi$ vertex



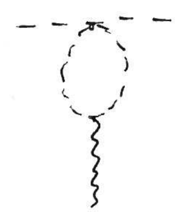
$D = 4+1-4 = 1$



$4-3\cdot 2+3$

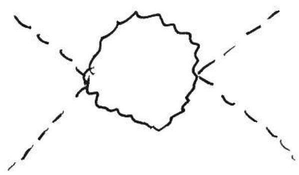


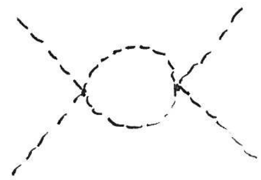
$4+1-4$



$4+1-4$

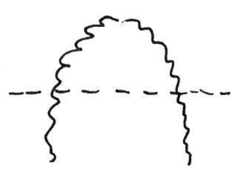
E=4 $\phi\phi\phi\phi$ vertex

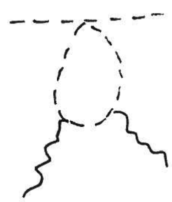


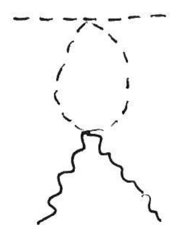


$D = 4-4 = 0$

$\phi\phi AA$ vertex







$D = 4-4 = 0$ $\quad$ $4+2-6$ $\quad$ $4-4$