NTNU Trondheim, Institutt for fysikk

Examination for FY3464 Quantum Field Theory I
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Allowed tools: mathematical tables
Feynman rules and some formulas can be found on p. 3.

1. Miscellaneous and quiz
(Several answers could be correct.)
a.) Evaluate (4 pts)

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu]$$

in four space-time dimensions.

b.) The field-strength of a Yang-Mills theory transforms under a local gauge transformation as: (1 pt)
☐ $\mathbf{F}(x) \to \mathbf{F}'(x) = \mathbf{F}(x)$
☐ $\mathbf{F}(x) \to \mathbf{F}'(x) = U(x)\mathbf{F}(x)U^\dagger(x)$
☐ $\mathbf{F}(x) \to \mathbf{F}'(x) = U(x)\mathbf{F}(x)U^\dagger(x) + \frac{\mathrm{i}}{g}(\partial_\mu U(x))U^\dagger(x)$
☐ $\mathbf{F}(x) \to \mathbf{F}'(x) = \mathbf{F}(x) + [D, \mathbf{F}(x)]$

c.) How many physical, how many unphysical degrees of freedom has the theory described by the Langrangian (3 pts)

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

with $F_{\mu\nu} = F^a_{\mu\nu} T^a$ and $T^a = \sigma^a/2$ (σ^a are the Pauli matrices)?

d.) How many physical, how many unphysical degrees of freedom has the theory

$$\mathscr{L}_{\mathrm{YM}} + \mathscr{L}_{\mathrm{FP}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \bar{c}^a \partial^\mu D^{ab}_\mu c^b \,,$$

i.e. adding a Faddev-Popov and a gauge-fixing term? (3 pts)

e.) The quantities c^a are: (2 pts)
☐ scalars
☐ fermions
☐ vector particles
☐ gauge fixing parameters
☐ bosons
☐ fermions.

f.) Explain in maximal three phrases why gravity has to be mediated by a spin $s = 2$ field (restriciting possible choices to $s \leq 2$). (3 pts)

a.) Contracting (2) with $\eta_{\mu\nu}$ gives

$$2\gamma^\mu\gamma_\mu = 2\eta^\mu_\mu = 8$$

or $\gamma^\mu\gamma_\mu = 4$. Together with $\text{tr}(\mathbf{1}) = 4$ we find

$$\text{tr}[\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu] = 2\eta^{\mu\nu}\gamma_\mu\gamma_\nu - \gamma^\nu\gamma^\mu\gamma_\mu\gamma_\nu = -2 \cdot 4 \cdot 4 = -32\,.$$

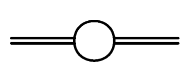
b.) The field-strength of a Yang-Mills theory transforms homogenously under a local gauge, $\mathbf{F}(x) \to \mathbf{F}'(x) = U(x)\mathbf{F}(x)U^\dagger(x)$.
c.) A massless spin-1 field has two spin degree of freedom, i.e. two out of four degrees of freedom in A_μ are unphysical. The three 2×2 Pauli matrices σ^a are the generators of SU(2) which agrees with the general result $N^2 - 1 = 3$. Each generator corresponds to one gauge boson, giving 3×2 physical and 3×2 unphysical degrees of freedom.
d.) Adding a gauge-fixing and a Faddev-Popov term does not change the physics: thus there are still 3×2 physical degrees of freedom. The six additional unphysical d.o.f. of the ghost fields $(c^1, \ldots, \bar{c}^3)$ compensate now explictely the unphysical time-like and longitudinal components of the gauge field.
e.) Ghost fields are fermionic scalars (Grassmann variable with no Lorentz index).
f.) A macroscopic force requires a bosonic mediator. Spin $s = 1$ leads to repulsion between masses, $s = 0$ and $s = 2$ to the desired attraction. In scalar gravity, the source is T^μ_μ which is zero for photons. Thus there would be no deflection of light and the equivalence principle would be violated, in contradiction to observations.

2. Scattering and decay of scalar fields.

Consider the theory of two light scalar fields ϕ_1 and ϕ_2 with mass m coupled to one heavy scalar Φ with mass $M > 2m$,

$$\mathscr{L} = \mathscr{L}_0 + g\phi_1\phi_2\Phi$$

where $\mathscr{L}_0$ is the free Lagrangian.
a.) Calculate the width Γ of the decay $\Phi \to \phi_1\phi_2$. (4 pts)
b.) Draw the Feynman diagram(s) and write down the Feynman amplitude $\mathrm{i}\mathcal{M}$ for the scattering process $\phi_1(p_1)\phi_2(p_2) \to \phi_1(p_1')\phi_2(p_2')$. What is your interpretation of the behaviour of the amplitude for $s = (p_1 + p_2)^2 \to M^2$? (5 pts)
c.) Consider the one-loop correction $\mathrm{i}\mathcal{M}_{\text{loop}}$ to the mass of Φ,

.

Write down $\mathrm{i}\mathcal{M}_{\text{loop}}$ first for an arbitrary momentum p of the external particle Φ, then for its rest frame, $p = (M, 0)$. Find the poles of the integrand and use the theorem of residues to perform the q^0 part of the loop integral. Finally, use the identity

$$\frac{1}{x \pm \mathrm{i}\varepsilon} = P\left(\frac{1}{x}\right) \mp \mathrm{i}\pi\delta(x)$$

to find the imaginary part of the amplitude. (8 pts)
Test: You should find $M\Gamma(\Phi \to \phi_1\phi_2) = \mathrm{Im}\mathcal{M}_{\mathrm{loop}}$, a special case of the optical theorem.
d.) What is the dimension of the coupling constant g? (2 pts)

a. The Feynman amplitude for the decay is $\mathrm{i}\mathcal{M} = -\mathrm{i}g$. Thus the angular integration gives simply 4π. In the rest frame of the decaying particle, $M^2 = 4(m^2 + p_{\mathrm{cms}}^2)$. Combined we find

$$\Gamma = \frac{g^2}{2M}\frac{1}{32\pi^2}\sqrt{1-\frac{4m^2}{M^2}}\,4\pi = \frac{g^2}{16\pi M}\sqrt{1-\frac{4m^2}{M^2}}\,.$$

b. The scattering amplitude consists of the s and the u channel exchange of the heavy scalar Φ,

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}g)^2\left[\frac{\mathrm{i}}{s-M^2+\mathrm{i}\varepsilon} + \frac{\mathrm{i}}{u-M^2+\mathrm{i}\varepsilon}\right]$$

with $s = (p_1+p_2)^2$ and $u = (p_2'-p_1)^2$. The second denominator never vanishes, while the first is zero for $s = M^2$, i.e. when the virtual scalar Φ is created on-shell. If we do not take the finite life-time of the heavy particle into account, it can travel (as a real particle) for infinite time, leading to an infinite range of the interaction.

c. The Feynman rules give

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}g)^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2+\mathrm{i}\varepsilon}\frac{\mathrm{i}}{(q-p)^2-m^2+\mathrm{i}\varepsilon}$$

Setting $p = (M,0)$ and $E_q = +\sqrt{q^2+m^2}$, we find as poles of the integrand $q^0 = E_q - \mathrm{i}\varepsilon$, $q^0 = -E_q + \mathrm{i}\varepsilon$, $q^0 = M + E_q - \mathrm{i}\varepsilon$, and $q^0 = M - E_q + \mathrm{i}\varepsilon$. We can choose the integration contour either in the upper or lower half-plane. Choosing the lower one, we pick up the two residues at $q^0 = E_q - \mathrm{i}\varepsilon$ and $q^0 = M + E_q - \mathrm{i}\varepsilon$. Hence we obtain

$$\mathcal{M} = -g^2\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2ME_q}\left(\frac{1}{M-2E_q+\mathrm{i}\varepsilon} + \frac{1}{M+2E_q-\mathrm{i}\varepsilon}\right)$$

The second denominator never vanishes and thus gives no contribution to the imaginary part. For the first one, we obtain using the given identity

$$\mathrm{Im}\mathcal{M} = g^2\pi\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{2ME_q}\delta(M-2E_q)$$

As $E_q = +\sqrt{q^2+m^2} \geq m$, the argument of the delta function is never zero for $M \leq 2m$ and the imaginary part of the amplitude vanishes thus. For $M > 2m$, we can perform the integral,

$$\mathrm{Im}\mathcal{M} = \frac{g^2}{16\pi}\sqrt{1-\frac{4m^2}{M^2}}$$

Thus we confirmed the relation $M\Gamma = \mathrm{Im}\mathcal{M}$.

d. The action $S = \int \mathrm{d}^4x \mathscr{L}$ is dimensionless, $(\partial_\mu\phi)^2$ implies then that scalar fields have mass dimension one in $D = 4$. Thus $[g] = 1$.

3. Vertex function

a.) Write down the most general form Λ^μ of the coupling term $\bar{u}(p')\Lambda^\mu u(p)$ between an external electromagnetic field and an on-shell Dirac fermion, consistent with Poincaré invariance, current conservation and parity. (6 pts)
b) Derive the Gordon decomposition,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{\mathrm{i}\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p)\,. \quad (1)$$

and use it to eliminate one of the three arbitrary functions in Λ^μ. (4 pts)
c.) Argue briefly for which values the two remaining functions are finite or diverge, assuming that the interaction is renormalisable. (3 pts)

a. Since $p^2 = p'^2 = m^2$, the only non-trivial scalar variable in the problem is $p\cdot p'$ or, equivalently, $q^2 = (p-p')^2$ as the variable on which the arbitrary scalar functions depend. Parity forbids the use of γ^5. Hence we use as ansatz

$$\Lambda^\mu(p,p') = A(q^2)\gamma^\mu + B(q^2)p^\mu + C(q^2)p'^\mu + D(q^2)\sigma^{\mu\nu}p_\nu + E(q^2)\sigma^{\mu\nu}p'_\nu\,.$$

Current conservation requires $q_\mu\Lambda^\mu(p,p') = 0$ and leads to $C = B$ and $E = -D$. Hence

$$\Lambda^\mu(p,p') = A(q^2)\gamma^\mu + B(q^2)(p^\mu + p'^\mu) + D(q^2)\sigma^{\mu\nu}q_\nu\,.$$

b. Evaluate

$$F^\mu = \bar{u}(p')\left[\not{p}'\gamma^\mu + \gamma^\mu\not{p}\right]u(p)$$

first using the Dirac equation for the two on-shell spinors, finding $F^\mu = 2m\bar{u}(p')\gamma^\mu u(p)$. Second, use $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu}$, obtaining

$$F^\mu = \bar{u}(p')\left[(p'+p)^\mu + \mathrm{i}\sigma^{\mu\nu}(p'-p)_\nu\right]u(p)\,,$$

and equate the two expressions. Uing also standard notation for the form factors, we can write

$$\begin{aligned}\Lambda^\mu(p,p') &= F_1(q^2)\gamma^\mu + F_2(q^2)\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m} = \\ &= F_1(q^2)\frac{(p'+p)^\mu}{2m} + [F_1(q^2)+F_2(q^2)]\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\,.\end{aligned}$$

c. We see that $F_1(q^2)$ is the coefficient of the electric charge. As this interaction is present in the original Langrangian of QED, $F_1(0)$ may (and does) diverge. Splitting $F_1(q^2)$ in an on-shell

and off-shell part corresponds to a Taylor expansion in the external momentum q^2, leading to additional powers of q^2 in the denominator. Thus the off-shell part is convergent.
The formfactor $F_2(q^2)$ corresponds to an interaction not present in the original Langrangian of QED and has to be finite.

Feynman rules and useful formulas

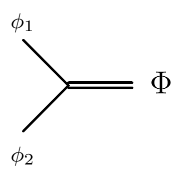
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,. \tag{2}$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{3}$$

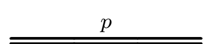
$$G(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} Z[J]\bigg|_{J=0}\,. \tag{4}$$

$$\mathcal{G}(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} \mathrm{i}W[J]\bigg|_{J=0}\,. \tag{5}$$

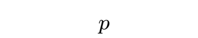
$$Z[J] = Z[0] \exp(\mathrm{i}W[J]) \tag{6}$$



$-\,\mathrm{i}g$



$$\frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\varepsilon}$$



$$\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2E_i}\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi^{(n)}\,. \tag{7}$$

The two particle phase space $\mathrm{d}\Phi^{(2)}$ in the rest frame of the decaying particle is

$$\mathrm{d}\Phi^{(2)} = \frac{1}{16\pi^2}\,\frac{|p'_{\mathrm{cms}}|}{M}\,\mathrm{d}\Omega\,, \tag{8}$$