Motivation.
We calculated the contribution of a scalar field with mass m to the vacuum energy density ρ twice, once introducing a cutoff M in momentum space and once using dimensional regularisation (DR). The two regularisation methods gave different results: The latter predicted $\rho \propto m^4$, while the former predicted $\rho \propto M^4$. Aim of the home exam is to obtain a better understanding why this happens and to decide which result is correct.

1. Energy-momentum stress tensor and E.o.S of the vacuum.
a.) Determine the energy-momentum stress tensor $T^{\mu\nu}$ of the free, real scalar field with Lagrange density

$$\mathscr{L} = \frac{1}{2}\eta_{\mu\nu}\left(\partial^\mu\phi\right)\left(\partial^\nu\phi\right) - \frac{1}{2}m^2\phi^2 - \rho_0$$

and show that the vacuum energy density ρ_0 acts like a cosmological constant,

$$T^{\mu\nu} = \eta^{\mu\nu}\rho_\Lambda\,.$$

b.) Find the energy-momentum stress tensor $T^{\mu\nu}$ for a perfect fluid in its rest frame (you can use the literature); compare the two stress tensors and show that the vacuum energy density ρ_0 and the cosmological constant have the equation of state (E.o.S.) $w = P/\rho = -1$ where P denotes the pressure.

2. Momentum cutoff.
Consider again a scalar field with mass m and keep the mass dependence throughout.
a.) Repeat the calculation leading to $\langle\rho\rangle \propto M^4$ [lecture notes (2.56-59)].
b.) Calculate in the same way the contribution of zero-point fluctuations to the pressure $\langle P\rangle$ of the vacuum. [If you are unfamiliar with the definition of pressure in kinetic theory, then you can deduce the required connection by comparing the expressions (2,3) with the one you used in a.).]
c.) Check if the E.o.E. $w = \langle P\rangle/\langle\rho\rangle = -1$ for a cosmological constant is satisfied for finite values of the cutoff M, i) for the leading M^4 terms, ii) for the subleading m^4 terms.

3. Dimensional regularisation.
Redo the calculations of 2.) applying now DR, i.e. calculate

$$\langle\rho\rangle = \frac{\mu^{4-d}}{(2\pi)^{(d-1)}}\int \mathrm{d}^{d-1}k\,\frac{\omega_{\boldsymbol{k}}}{2}$$

and $\langle P\rangle$ for $d = 4 - \varepsilon$. (Find the d dimensional surface integral and express the remaining integral as Beta function.) Check again the E.o.S. w. Separate the expression for $\langle\rho\rangle$ into a finite and a pole part.

4. Interpretation.
What is your interpretation of the results obtained (less than 100 words)?

Useful relations.
The number density n, energy density ρ and pressure P of a species X follows as

$$n = \frac{g}{(2\pi)^3} \int d^3p \, f(p) \tag{1}$$

$$\rho = \frac{g}{(2\pi)^3} \int d^3p \, E f(p) \tag{2}$$

$$P = \frac{g}{(2\pi)^3} \int d^3p \, \frac{p^2}{3E} \, f(p) \tag{3}$$

where the factor g takes into account the internal degrees of freedom like spin or colour.

The surface Ω_d of a unit sphere in d dimensions is given by $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$.

Euler's beta function is defined by

$$B(a,b) \equiv \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^\infty \mathrm{d}t \, \frac{t^{a-1}}{(1+t)^{a+b}} \, . \tag{4}$$