NTNU Trondheim, Institutt for fysikk

Examination for FY3464 Quantum Field Theory I
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Allowed tools: mathematical tables
Feynman rules and some formulas can be found on p. 3.

1. Miscellaneous and quiz
(Several answers could be correct.)
a.) Write down A^* for (4 pts)

$$A = \bar{u}(p_2)\gamma^\mu u(p_1)$$

Starting from

$$A^* = A^\dagger = (u^\dagger(p_2)\gamma^0\gamma^\mu u(p_1))^\dagger = u^\dagger(p_1)\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_2),$$

and using $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ and $(\gamma^0)^2 = 1$, we arrive at

$$A^* = \bar{u}(p_1)\gamma^\mu u(p_2).$$

b.) The covariant derivative of a Yang-Mills theory transforms under a local gauge transformation $U(x)$ as: (2 pts)

☐ $D_\mu \to D'_\mu = D_\mu$
☐ $D_\mu \to D'_\mu = U(x)D_\mu U^\dagger(x)$
☐ $D_\mu \to D'_\mu = U(x)D_\mu U^\dagger(x) + \frac{i}{g}(\partial_\mu U(x))U^\dagger(x)$
☐ $D_\mu \to D'_\mu = D_\mu + [F_{\mu\nu}, D^\nu]$

The covariant derivative should satisfy $D_\mu\psi(x) \to [D_\mu\psi(x)]' = U(x)[D_\mu\psi(x)]$ for $\psi(x) \to \psi'(x) = U(x)\psi(x)$. Combining both equations gives

$$D_\mu\psi(x) \to [D_\mu\psi]' = UD_\mu\psi = UD_\mu U^{-1}U\psi = UD_\mu U^{-1}\psi',$$

and thus the covariant derivative transforms homogenously, $D'_\mu = UD_\mu U^{-1}$.

c.) A fermionic propagator $S_F(x - x')$ is: (2 pts)

☐ an even function of distance $x - x'$.
☐ an odd function of distance $x - x'$.

A bosonic propagator is:

☐ an even function of distance $x - x'$.
☐ an odd function of distance $x - x'$.

Bosonic fields should commute, fermionic fields anticommute. Thus the propagtor (= 2-point function) of a boson is even, of a fermion odd. This comes out automatically, since bosons (fermions) satisfy 2.nd (1.st) order wave equations, leading to a polarisation sum in the propagator which is quadratic (linear) in the momentum.

d.) A Faddev-Popov ghost is a: (2 pts)
☐ spin $s = 0$ particle
☐ spin $s = 1/2$ particle
☐ spin $s = 1$ particle
☐ gauge fixing parameter
☐ boson
☐ fermion.

A fermionic spin $s = 0$ particle.

e.) QCD is a Yang-Mills theory with gauge group SU(3). How many degrees of freedom have the Faddev-Popov ghosts in QCD? (2 pts)

The $N^2 - 1 = 8$ generators of SU(3) correspond to 8 massless gluons. A massless spin-1 field has two spin degree of freedom, i.e. two (time-like and longitudinal) out of four degrees of freedom in A_μ are unphysical. The Faddev-Popov ghosts are introduced to cancel these 16 unphysical degrees of freedom: the ghost live as the gluons in the same (adjoint) representation of SU(3), $c^a, \bar{c}^a$, $a = 1..8$, summing up to 16 ghosts.

2. The $\lambda\phi^3$ theory.
Consider the theory of a real scalar field ϕ with mass m and a $\frac{\lambda}{3!}\phi^3$ self-interaction in $d = 6$ dimensions.
a.) Write down the Lagrange density $\mathscr{L}$ and explain your choice of signs. (6 pts)
b.) Write down the corresponding generating functional for disconnected Green functions. (3 pts)
c.) Determine the dimension of the field ϕ and of the coupling λ. (3 pts)
d.) Draw the Feynman diagram(s) and write down the analytical expression for the self-energy iΣ (i.e. the loop correction for the free propgator) at order $\mathcal{O}(\lambda^2)$ in momentum space. (4 pts)
e.) Determine the symmetry factor of iΣ. (3 pts)
f.) Calculate the self-energy iΣ using dimensional regularisation, split the result into divergent pole terms and finite reminder. (16 pts)
g.) Draw the primitive divergent diagrams in this theory and determine their superficial degree of divergence. [primitive divergent = divergent 1PI one-loop graphs] (6 pts)

a.) The free Lagrangian is

$$\mathscr{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

the relative sign is fixed by the relativistic energy-momentum relation, the overall sign by the requirement that the Hamiltonian is bounded from below. As the self-interaction is odd, adding $+\frac{\lambda}{3!}\phi^3$ or $-\frac{\lambda}{3!}\phi^3$ is equivalent: both choices will lead to an unstable vacuum. In order to reproduce the Feynman rule, we should choose $\mathscr{L}_I = -\frac{\lambda}{3!}\phi^3$.

b.) The generating functional $Z[J]$ of disconnected Green functions is obtained from the path integral by i) adding a linear coupling to an external source J, ii) taking the limit $t, -t' \to \infty$ with $m^2 - \mathrm{i}\varepsilon$,

$$Z[J] = \langle 0|0\rangle_J = \mathcal{N} \int \mathcal{D}\phi \exp \mathrm{i} \int_\Omega \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3 + J\phi \right) .$$

c.) The action $S = \int \mathrm{d}^6x\mathscr{L}$ has to be dimensionless. Thus $[\mathscr{L}] = m^6$, $[\phi] = m^2$, and thus the coupling is dimensionless, $[\lambda] = m^0$. [That's the reason why we do this exercise in $d = 6$.]

Using the Feynman rules gives for

in momentum space

$$\mathrm{i}\Sigma(k^2) = S\,(-\mathrm{i}\lambda)^2 \int \frac{\mathrm{d}^6p}{(2\pi)^6} \frac{\mathrm{i}}{(p+k)^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

where the symmetry factor S is determined in e.) and the vertex $-\mathrm{i}\lambda$ was used.

e.) The self-energy is a second order diagram, corresponding to the term

$$\frac{1}{2!}\left(\frac{-\mathrm{i}\lambda}{3!}\right)^2 \int \mathrm{d}^4y_1\mathrm{d}^4y_2 \langle 0|T[\phi(x_1)\phi(x_2)\phi^3(y_1)\phi^3(y_2) + (y_1 \leftrightarrow y_2)$$

in the perturbative expansion in coordinate space. The exchange graph $y_1 \leftrightarrow y_2$ is identical to the original one, canceling the factor 1/2! from the Taylor expansion. We count the number of possible ways to combine the fields in the time-ordered product into four propagators. We have three possibilities to contract $\phi(x_1)$ with a $\phi(y_1)$. Similiarly, there are three possibilities for $\phi(x_2)\phi(y_2)$. The remaining pairs of $\phi(y_1)$ and $\phi(y_2)$ can be contracted in 2! ways. Thus the symmetry factor is

$$S = \left(\frac{1}{2!} \times 2\right)\left(\frac{1}{3!}\right)^2 (3 \times 3 \times 2!) = \frac{1}{2}$$

[The symmetry factor is given for the vertex $-\mathrm{i}\lambda$.]

f.) We combine the two propagators (suppressing the $\mathrm{i}\varepsilon$) using (9),

$$\frac{1}{(p+k)^2 - m^2} \frac{1}{p^2 - m^2} = \int_0^1 \mathrm{d}x \frac{1}{D^2}$$

with

$$\begin{aligned} D &= x[(p+k)^2 - m^2] + (1-x)(p^2 - m^2) \\ &= (p+xk)^2 + x(1-x)k^2 - m^2 = q^2 + f\,, \end{aligned}$$

where we introduced $q = p + xk$ as new integration variable and set $f = x(1-x)k^2 - m^2$. We go now to $d = 2\omega = 6 - \varepsilon$ dimensions,

$$\mathrm{i}\Sigma(k^2) = \frac{1}{2}\lambda^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q+f)^2}.$$

Evaluating the integral with (10), using $\Gamma(2) = 1$ and $\omega = 3 - \varepsilon/2$ gives

$$\Sigma(k^2) = -\frac{\lambda^2}{2} \frac{\Gamma(-1+\varepsilon/2)}{(4\pi)^3} \int_0^1 \mathrm{d}x f \left(\frac{4\pi\mu^2}{f}\right)^{\varepsilon/2}.$$

Here, we added a mass scale μ in order to make the ε dependent term dimensionless such that we can expand it using (11),

$$\left(\frac{4\pi\mu^2}{f}\right)^{\varepsilon/2} = 1 + \frac{\varepsilon}{2}\ln\left(\frac{4\pi\mu^2}{f}\right) + \mathcal{O}(\varepsilon^2).$$

Expanding also

$$\Gamma(-1+\varepsilon/2) = -\left[\frac{2}{\varepsilon} + 1 - \gamma + \mathcal{O}(\varepsilon)\right]$$

we arrive at

$$\Sigma(k^2) = \frac{\alpha}{2}\left[\left(\frac{2}{\varepsilon} + 1 - \gamma\right)\left(\frac{k^2}{6} - m^2\right) + \int_0^1 \mathrm{d}x f \ln\left(\frac{4\pi\mu^2}{f}\right)\right]$$

where we used $\int_0^1 \mathrm{d}x f = k^2/6 - m^2$ and set $\alpha = \lambda^2/(4\pi)^3$. The obtained expression for the self-energy has the UV divergence isolated into an $1/\varepsilon$ pole which is ready for subtraction.

g.) The primitive divergent diagrams are the divergent 1PI 1-loop diagrams. We can order them by the number E of external legs and determine the superficial degree of divergence D by naive power-counting,

$$\int \mathrm{d}^6 p (p^{-2})^I \sim \int^\Lambda \mathrm{d}p p^5 (p^{-2})^I \sim \Lambda^D$$

where I is the number of internal lines; see the last page for the Feynman diagrams.
$E = 0$ and $D = 6$ corresponding a contribution to the cosmological constant,
$E = 1$ and $D = 4$ corresponding to a tadpole diagram,
$E = 2$ and $D = 2$ corresponding to the self-energy.
$E = 3$ and $D = 0$ corresponding to the vertex correction.
(The vacuum graph $E = 0$ is optional – you may prefer to "hide" them by asking for a properly normalized generating functional.)

3. Spin-1 propagator
a.) Use the tensor method to determine the propagator $D_{\mu\nu}(k)$ of a massive spin-1 field described by the Proca equation (8 pts)

$$(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)\,A_\nu + m^2 A^\mu = 0\,.$$

b) Give one argument why this method does not work for $m = 0$. (2 pts)

a.) We write frist $m^2 A^\mu = m^2\eta^{\mu\nu}A_\nu$. The propagator $D_{\mu\nu}$ for a massive spin-1 field is determined by

$$\left[\eta^{\mu\nu}(\Box + m^2) - \partial^\mu\partial^\nu\right] D_{\nu\lambda}(x) = \delta^\mu_\lambda\delta(x)\,. \tag{1}$$

Inserting the Fourier transformation of the propagator and the delta function gives

$$\left[\left(-k^2 + m^2\right)\eta^{\mu\nu} + k^\mu k^\nu\right] D_{\nu\lambda}(k) = \delta^\mu_\lambda\,. \tag{2}$$

We will apply the tensor method to solve this equation: In this approach, we use first all tensors available in the problem to construct the required tensor of rank 2. In the case at hand, we have at our disposal only the momentum k_μ of the particle—which we can combine to $k_\mu k_\nu$—and the metric tensor $\eta_{\mu\nu}$. Thus the tensor structure of $D_{\mu\nu}(k)$ has to be of the form

$$D_{\mu\nu}(k) = A\eta_{\mu\nu} + Bk_\mu k_\nu \tag{3}$$

with two unknown scalar functions $A(k^2)$ and $B(k^2)$. Inserting this ansatz and multiplying out, we obtain

$$\begin{aligned}\left[(-k^2 + m^2)\eta^{\mu\nu} + k^\mu k^\nu\right]\left[A\eta_{\nu\lambda} + Bk_\nu k_\lambda\right] &= \delta^\mu_\lambda\,,\\ -Ak^2\delta^\mu_\lambda + Am^2\delta^\mu_\lambda + Ak^\mu k_\lambda + Bm^2 k^\mu k_\lambda &= \delta^\mu_\lambda\,,\\ -A(k^2 - m^2)\delta^\mu_\lambda + (A + Bm^2)k^\mu k_\lambda &= \delta^\mu_\lambda\,. \end{aligned} \tag{4}$$

In the last step, we regrouped the LHS into the two tensor structures δ^μ_λ and $k^\mu k_\lambda$. A comparison of their coefficients gives then $A = -1/(k^2 - m^2)$ and

$$B = -\frac{A}{m^2} = \frac{1}{m^2(k^2 - m^2)}.$$

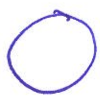
Thus the massive spin-1 propagator follows as

$$D_F^{\mu\nu}(k) = \frac{-\eta^{\mu\nu} + k^\mu k^\nu/m^2}{k^2 - m^2 + \mathrm{i}\varepsilon}\,. \tag{5}$$

b.) There's a mismatch of degrees of freedom, $3 \leftrightarrow 2$, between the massive and massless case/The longitudinal part $k^\mu k^\nu/m^2$ which blows up for $m \to 0$ does not contribute to the massless propagator/The projection operator in (5) has an eigenvalue 0 and is thus not invertible.

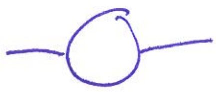
Primitive divergent diagrams for 2.g):

$E=0,\ \mathcal{D}=6$

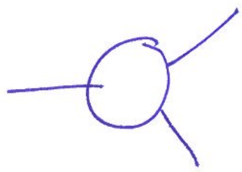


$E=1,\ \mathcal{D}=4$

$E=2,\ \mathcal{D}=2$



$E=3,\ \mathcal{D}=0$



I used a non-standard choice of the Feynman rule which is normally given as $-i\lambda$. Possible confusion caused by this was interpreted in your favor.

Some formulas

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \,. \tag{6}$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{7}$$

$$\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0 \tag{8}$$

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1-z)]^2} \,. \tag{9}$$

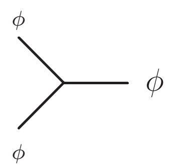
$$\begin{aligned} I(\omega,\alpha) &= \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 + 2pk + M^2 + \mathrm{i}\varepsilon]^\alpha} \\ &= \mathrm{i}\, \frac{(-\pi)^\omega}{(2\pi)^{2\omega}} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)} \frac{1}{[M^2 - p^2 + \mathrm{i}\varepsilon]^{\alpha-\omega}} \,. \end{aligned} \tag{10}$$

$$f^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2} \ln f + \mathcal{O}(\varepsilon^2) \,. \tag{11}$$

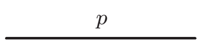
$$\Gamma(n+1) = n! \tag{12}$$

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \psi_1(n+1) + \mathcal{O}(\varepsilon)\right] \,, \tag{13}$$

$$\psi_1(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma \,, \tag{14}$$



$-\,\mathrm{i}\lambda$



$$\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$$