NTNU Trondheim, Institutt for fysikk

Examination for FY3464 Quantum Field Theory I
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Allowed tools: mathematical tables

1. Spin zero.
Consider a real, scalar field ϕ with mass m and a quartic self-interaction proportional to λ in $d = 4$ space-time dimensions. (4 pts)
a.) Write down the Lagrange density $\mathscr{L}$, explain your choice of signs and pre-factors (when physically relevant). (4 pts)
b.) Determine the mass dimension of all quantities in the Lagrange density $\mathscr{L}$. (6 pts)
c.) Draw the Feynman diagrams for $\phi\phi \to \phi\phi$ scattering at $\mathcal{O}(\lambda^2)$, determine the symmetry factor of these diagrams, and write down the expression for the Feynman amplitude $\mathrm{i}\mathcal{A}$ of this process in momentum space. (8 pts)
d.) The one loop correction to the scalar propagator is

$$G^{(2)}(p) = \frac{\mathrm{i}}{p^2 - m^2 - \frac{\mathrm{i}\lambda}{2}\Delta_F(0) + \mathrm{i}\varepsilon}\,. \tag{1}$$

Calculate the self-energy or mass correction $\delta m^2 = \frac{\mathrm{i}\lambda}{2}\Delta_F(0)$ in dimensional regularisation (DR). You should end up with something of the form (10 pts)

$$\delta m^2 = \lambda m^2[a/\varepsilon + b + c\ln(\mu^2/m^2)]\,. \tag{2}$$

e.) What is your interpretation of the dependence of δm^2 on the parameter μ in Eq. (2)? [max. 50 words or one formula without explicit calculation is enough] (4 pts)

2. Spin one-half. Consider a theory of two Weyl fields, a left-chiral field ϕ_L and a right-chiral field ϕ_R, with kinetic energy

$$\mathscr{L}_0 = \mathrm{i}\phi_R^\dagger \sigma^\mu \partial_\mu \phi_R + \mathrm{i}\phi_L^\dagger \bar{\sigma}^\mu \partial_\mu \phi_L \tag{3}$$

a.) Add a Dirac mass term $\mathscr{L}_D$. (3 pts)
b.) Find the transformation property of $\mathscr{L}_0$ and $\mathscr{L}_D$ under parity, $P\boldsymbol{x} = -\boldsymbol{x}$. (4 pts)
c.) Add a coupling $\mathscr{L}_{\text{int}}$ to the photon A^μ such that the coupling constant is dimensionless. (4 pts)

3. Spin one.
Consider a massless spin-one particle, e.g. the photon A^μ with Lagrange density

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{cl}} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\,. \tag{4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
a.) List the symmetries of $\mathscr{L}_{\rm cl}$, and of $\mathscr{L}_{\rm eff}$. (5 pts)
b.) Derive the corresponding propagator $D_{\mu\nu}(k)$. [You don't have to care how the pole is handled.] (10 pts)
c.) Write down the generating functionals for disconnected and connected Green functions of this theory. (4 pts)
d.) How does one obtain connected Green functions from the generating functional? (3 pts)
e.) What are the two main changes in $\mathscr{L}_{\rm cl}$ and in $\mathscr{L}_{\rm eff} - \mathscr{L}_{\rm cl}$ in case of a non-abelian theory? [max. 50 words] (4 pts)

Some formulas

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

They satisfy $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\varepsilon^{ijk}\sigma^k$. Combining the Pauli matrices with the unit matrix, we can construct the two 4-vectors $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$.

The Gamma matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{6}$$

and are in the Weyl or chiral representation given by

$$\gamma^0 = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{7}$$

$$\gamma^i = \sigma^i \otimes \mathrm{i}\tau_3 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{8}$$

$$\gamma^5 = 1 \otimes \tau_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9}$$

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \equiv P_L\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1+\gamma^5)\psi \equiv P_R\psi\,. \tag{10}$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{11}$$

$$\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0 \tag{12}$$

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1-z)]^2} \,. \tag{13}$$

$$\frac{1}{k^2 + m^2} = \int_0^\infty \mathrm{d}s \, \mathrm{e}^{-s(k^2+m^2)} \tag{14}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp(-x^2/2) = \sqrt{2\pi} \tag{15}$$

$$f^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2} \ln f + \mathcal{O}(\varepsilon^2) \,. \tag{16}$$

$$\Gamma(z) = \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-t} t^{z-1} \tag{17}$$

$$\Gamma(n+1) = n! \tag{18}$$

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\varepsilon} + \psi_1(n+1) + \mathcal{O}(\varepsilon) \right] , \tag{19}$$

$$\psi_1(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma \,, \tag{20}$$