# NTNU Trondheim, Institutt for fysikk

**Examination for FY3464 Quantum Field Theory I**
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Allowed tools: mathematical tables

**1. Noether's theorem.**
Show that a continuous global symmetry of a set of fields $\phi_a$ described by a Lagrangian $\mathscr{L}(\phi_a, \partial_\mu \phi_a)$ leads classically to the conserved current

$$j^\mu = \frac{\delta \mathscr{L}}{\delta \partial_\mu \phi_a} \, \delta\phi_a - K^\mu \,, \tag{1}$$

where $K^\mu$ is a four-divergence, $\delta\mathscr{L} = \partial_\mu K^\mu$. (6 pts)

**2. A complex scalar field.**
Consider a complex, scalar field $\phi$ with mass $m$ and a quartic self-interaction proportional to $\lambda$ in $d = 4$ space-time dimensions.
a.) Write down its Lagrange density $\mathscr{L}_s$, explain your choice of signs and pre-factors (when physically relevant). (6 pts)
b.) Determine the mass dimension of all quantities in the Lagrange density $\mathscr{L}_s$. (6 pts)
c.) Show that the Lagrange density $\mathscr{L}_s$ is invariant under global phase transformations and determine the conserved current $j^\mu$. (6 pts)

**3. Scalar QED.**
Consider now the complex, scalar field $\phi$ coupled to the photon $A^\mu$, i.e. a massless spin-1 field which field-strength satisfies $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
a.) Find the coupling $\mathscr{L}_I$ between $\phi$ and $A^\mu$ requiring that $\mathscr{L}_s + \mathscr{L}_I$ is invariant under local phase transformations; determine the transformation law for $A_\mu$. (10 pts)
b.) Write down the generating functionals for disconnected and connected Green functions of this theory. (6 pts)
c.) How does one obtain connected Green functions for the photon from the generating functional? (4 pts)
d.) Find the Feynman rules for the vertices involving photons and scalars. (6 pts)
e.) Define the superficial degree of divergence $D$ and draw for each of the cases $D = \{0, 1, 2\}$ one 1-loop Feynman diagram. (7 pts)

**4. Tensor decomposition.**
Consider the *real* decay process $\mu \to e + \gamma$ in Minkowski space, allowing for parity violation.

Write its Lorentz invariant amplitude $\mathcal{A}$ as $\mathcal{A}(\mu \to e+\gamma) = \varepsilon_\lambda \langle e| J^\lambda_{\text{em}} |\mu\rangle$ where $J^\lambda_{\text{em}}$ denotes the electromagnetic current and decompose it in scalar functions $A, B, \ldots$ as

$$\langle e| J^\lambda_{\text{em}} |\mu\rangle = \bar{u}_e(p')[A\gamma^\lambda + \ldots]u_\mu(p)\,.$$

Use the symmetries to express $\mathcal{A}$ by the minimal number of scalar functions required. [Note the difference to the treatment of the electromagnetic vertex in the lectures where the photon was virtual.] (10 pts)

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Some formulas

The Gamma matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{2}$$

and are in the Weyl or chiral representation given by

$$\gamma^0 = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

$$\gamma^i = \sigma^i \otimes \mathrm{i}\tau_3 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{4}$$

$$\gamma^5 = 1 \otimes \tau_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \equiv P_L\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1+\gamma^5)\psi \equiv P_R\psi\,. \tag{6}$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{7}$$