Relativistisk kvantemekanikk 12.6.93

Løsninger

1a **Kanonisk form:** Lineariserer $E^2 = p^2c^2 + (mc^2)^2$ og bruker "minimal kobling" $\vec{p} \to \vec{p} - e\vec{A}$

$$H\psi = [c\,\vec{\alpha}(\vec{p} - e\vec{A}) + \beta mc^2 + e\varphi]\psi = E\psi \quad \text{med} \quad \vec{p} = -i\hbar\nabla, \quad E = i\hbar\frac{\partial}{\partial t}$$

De hermiteske Dirac matrisene må oppfylle antikommuterings reglene

$$\{\alpha^i, \alpha^k\}_+ = 2\delta^{ik} \qquad \{\beta, \alpha^i\}_+ = 0 \qquad \beta^2 = 1 \qquad i,k = 1,2,3$$

I standardrepresentasjonen er 4×4 matriser gitt ved

$$\beta = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \qquad \alpha^k = \begin{pmatrix} & \sigma^k \\ \sigma^k & \end{pmatrix} \quad \text{hvor} \quad \sigma^k = \left\{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & -i \\ i & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right\} \text{ er } 2\times2 \text{ Paulimatr}$$

Spinor:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Kovariant form

Flytter over $E \to E - e\varphi$, multipliserer med β og setter $\beta\alpha^k = \gamma^k$, $\beta = \gamma^0$:

$$[\gamma^\mu(p_\mu - eA_\mu) - mc]\psi = 0 \qquad \text{med} \quad p_\mu = i\hbar\frac{\partial}{\partial x^\mu}, \quad A_\mu = \left(\tfrac{1}{c}\varphi, -\vec{A}\right)$$

Betingelser:

$$\{\gamma^\mu, \gamma^\nu\}_+ = 2g^{\mu\nu} \qquad \mu = 0,1,2,3 \quad \text{Metrikk:} \quad g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$\gamma^{0+} = \gamma^0$ (hermitesk) $\qquad \gamma^{k+} = -\gamma^k$ antihermitesk

For elektron $e = -|e|$ $\qquad [\gamma^\mu(p_\mu + |e|A_\mu) - mc]\psi_e = 0$

b) For positron $e = +|e|$ $\qquad [\gamma^\mu(p_\mu - |e|A_\mu) - mc]\psi_p = 0 \qquad p_\mu = i\hbar\frac{\partial}{\partial x^\mu}$

Tar kompleks konjugert og multipliserer med C (4×4 matrise)

$$C[\gamma^{\mu*}C^{-1}C\,(p_\mu^* - |e|A_\mu) - mc]\psi_p^* = 0 \qquad p_\mu^* = -i\hbar\frac{\partial}{\partial x^\mu}, \quad A_\mu \text{ reell}$$

$$[-C\gamma^{\mu*}C^{-1}(p_\mu + |e|A_\mu) - mc]\,C\psi_p^* = 0$$

Dette blir elektron likningen hvis $-C\gamma^{\mu*}C^{-1} = \gamma^\mu$ og $C\psi_p^* = \psi_e$

I standardrepresentasjonen:

$$\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} & \vec{\sigma} \\ -\vec{\sigma} & \end{pmatrix}$$

er $\gamma^{2*} = -\gamma^2$ imaginær, mens resten $\gamma^{0*} = \gamma^0, \gamma^{1*} = \gamma^1, \gamma^{3*} = \gamma^3$ reell.

Betingelsene blir da

$C\gamma^0 = -\gamma^0 C$ $\qquad$ C kommuterer med γ^2 og antikommuterer med de andre

$C\gamma^1 = -\gamma^1 C$ $\qquad C = a\gamma^2$

$C\gamma^2 = +\gamma^2 C$ $\qquad$ ψ_e blir normert hvis ψ_p er det når $a = e^{i\chi}$ (χ reell)

$C\gamma^3 = -\gamma^3 C$ $\qquad$ Velger $a = i$ blir C reell $\qquad \underline{C = i\gamma^2}$

c) Fri partikkel i ro: $\vec{p} = 0$, $p^0 = \frac{E}{c} = \frac{1}{c}(mc^2 + c^2(\vec{p})^2) = mc = p_0$, $\vec{A} = 0$, $\varphi = 0$

$$(c\gamma^0 p_0 - mc^2)\psi = 0 \qquad cp^0 = i\hbar\frac{\partial}{\partial t}$$

I standard representasjonen $\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\Bigg\} \Rightarrow$ $\begin{array}{l} \left(i\hbar\frac{\partial}{\partial t} - mc^2\right)\psi_a = 0 \\ \left(-i\hbar\frac{\partial}{\partial t} - mc^2\right)\psi_b = 0 \end{array}$ $\qquad \psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}$

Rel. kv. m 12.6.97

1c forts.

Løsning for $E = mc^2 > 0$ $\quad \psi_a = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = u_a \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}mc^2t}$, $\quad \psi_b = 0$ $\quad u_a = \begin{pmatrix}u_1\\ u_2\end{pmatrix}$ = konstant

for $E = -mc^2 < 0$ $\quad \psi_a = 0$ $\quad \psi_b = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = u_b \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{\frac{i}{\hbar}mc^2t}$ $\quad u_b = \begin{pmatrix}u_3\\ u_4\end{pmatrix}$ = konstant

4 løsninger:
$$\psi_0^{(i)} = u^{(i)}\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}E_0^{(i)}t} \qquad E_0^{(i)} = \begin{cases} mc^2 & \text{for } i = 1,2 \\ -mc^2 & i = 3,4 \end{cases} \qquad u^{(i)} = \begin{pmatrix}u_1^{(i)}\\ u_2^{(i)}\\ u_3^{(i)}\\ u_4^{(i)}\end{pmatrix}$$

med
$$u_0^{(1)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u_0^{(2)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad u_0^{(3)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad u_0^{(4)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$
Løsning (i) = (3) og (4) har negativ energi.

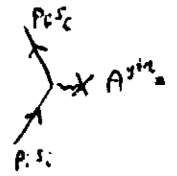
$u^{(1)}$ og $u^{(3)}$ har spinnkomponent opp langs z-aksen, $u^{(2)}$ og $u^{(4)}$ ned:
$$\Sigma^z u_0^{(3)} = \begin{pmatrix}\sigma^z & \\ & \sigma^z\end{pmatrix} u_0^{(3)} = \begin{pmatrix}1 &&& \\ & -1 && \\ && 1 & \\ &&& -1\end{pmatrix} u_0^{(3)} = \underline{+u_0^{(3)}}, \qquad \Sigma^z u_0^{(4)} = \underline{-u_0^{(4)}}$$

d)
$$\psi_p = (C^{-1}\psi_e)^* = i\gamma^2\psi_e^* \qquad (C^{-1} = i\gamma^2)$$

$$\psi_e^{(3)} \Rightarrow \psi_p^{(3)} = i\begin{pmatrix} &&& -i \\ && i & \\ & i && \\ -i &&& \end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}mc^2t} = \underline{\begin{pmatrix}0\\-1\\0\\0\end{pmatrix}\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}mc^2t} = -\psi_e^{(2)}}$$ Positiv energi og spinn ned

$$\psi_e^{(4)} \Rightarrow \psi_p^{(4)} = i\begin{pmatrix} &&& -i \\ && i & \\ & i && \\ -i &&& \end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}mc^2t} = \underline{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{-\frac{i}{\hbar}mc^2t} = \psi_e^{(1)}}$$ Positiv energi og spinn opp.

<u>Oppgave 2</u> a) Fra Feynman reglene $(\hbar = c = 1)$



$$S_{fi}^{(1)} = \left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{m}{E_f}\right)^{1/2} \bar{u}(p_f s_f)\left(-ie\gamma^\mu A_\mu^{ytre}(\vec{p}_f - \vec{p}_i)\, 2\pi\, \delta(E_f - E_i)\right)\left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{m}{E_i}\right)^{1/2} u(p_i s_i)$$
$$= \left(\frac{1}{2\pi}\right)^3 \frac{m}{\sqrt{E_f E_i}}(-ie)\left(\bar{u}_f\gamma^0 A_0 u_i\right) 2\pi\, \delta(E_f - E_i)$$

da $A = (A_0, 0)$ med $A_0(\vec{q}) = -\int \frac{Ze}{4\pi\varepsilon_0}\frac{1}{r} e^{i\vec{q}\vec{r}} d^3r = -\frac{Ze}{\varepsilon_0|\vec{q}|^2}$ $\quad \vec{q} = \vec{p}_f - \vec{p}_i$

Gir (med $\varepsilon_0 = 1$)
$$S_{fi}^{(1)} = \frac{Ze^2}{(2\pi)^3}\frac{m}{\sqrt{E_f E_i}}\frac{\bar{u}(p_f s_f)\gamma^0 u(p_i s_i)}{|\vec{p}_f - \vec{p}_i|^2}\, 2\pi i\, \delta(E_f - E_i) \equiv K_{fi}\, 2\pi\, \delta(E_f - E_i)$$

b) Antall overganger pr tidsenhet til tilstander i impulsintervallet d^3p_f
$$dw_{fi} = \frac{1}{T}|S_{fi}|^2 dn_f \qquad dn_f = \frac{V d^3p_f}{(2\pi\hbar)^3} = d^3p_f \text{ når } \hbar = 1 \text{ og her } V = (2\pi)^3$$

T = overgangstiden $(\to\infty)$ $\quad V$ = normeringsvolumet (her $= (2\pi)^3$)

Diff. spredningstverrsnitt
$$\frac{d\sigma}{d\Omega} = \frac{\text{antall spredte partikler pr tidsenhet og pr romvinkelenhet}}{\text{innfallende intensitet}}$$
$$= \frac{\frac{dw_{fi}}{d\Omega}}{\frac{v_i}{V}} = \frac{\frac{1}{T}|S_{fi}|^2\frac{dn_f}{d\Omega}}{v_i/(2\pi)^3} \qquad T\to\infty$$
$$= (2\pi)^4 |K_{fi}|^2 \frac{|\vec{p}_f|\, E_f\, dE_f}{v_i}\, \delta(E_f - E_i)$$

når en bruker $d^3p_f = |\vec{p}_f|^2 dp_f\, d\Omega = |\vec{p}_f|\, E_f\, dE_f\, d\Omega$

og $\left(\delta(E_f - E_i)\right)^2 = \lim_{T\to\infty}\left(\frac{1}{2\pi}\int_{-T/2}^{T/2} e^{i(E_f - E_i)t}\,dt\right) = \lim_{T\to\infty}\left(\frac{1}{2\pi}\int_{-T/2}^{T/2} dt \int_{-T/2+t}^{T/2+t} e^{i(E_f-E_i)(t+t')}\,d(t+t')\right)$

$$= \frac{T}{2\pi}\,\delta(E_f - E_i)$$

Ved integrasjon om E_i og benyttelse av $v_i = \frac{|\vec{p}_i|}{E_i}$

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \frac{4}{(2\pi)^4}\left(\frac{Ze^2}{4\pi}\right)^2 \int \frac{m^2}{E_i E_f}\, \frac{|\bar{u}_f \gamma^0 u_i|^2}{|\vec{q}|^4}\, \frac{E_i |\vec{p}_f|\, E_f\, \delta(E_f - E_i)\, dE_i}{|\vec{p}_i|} = \underline{\left(\frac{Ze^2}{4\pi}\right)^2 \frac{m^2}{|\vec{q}|^4}\left|\bar{u}_f \gamma^0 u_i\right|^2}$$

b ii) Dirac-likningen $(\not{p} - m)\psi = 0$ gir for

positiv-energi tilstand $u(p)\, e^{-ipx}$ $\quad (\not{p} - m)u = 0$

og for negativ-energi tilstand $v(p)\, e^{ipx}$ $\quad (\not{p} + m)v = 0$

Herav

$$P_+ u = \frac{m + \not{p}}{2m} u = \frac{m + m}{2m} u = u \qquad P_+ v = \frac{m + \not{p}}{2m} v = \frac{m - m}{2m} v = 0$$

$$P_- v = \frac{m - \not{p}}{2m} v = \frac{m + m}{2m} v = v \qquad P_- u = \frac{m - \not{p}}{2m} u = \frac{m - m}{2m} u = 0$$

Fra superposisjon av pos. og neg. energi tilstander projiserer altså P_+ ut de positive

$$P_+ \sum_s \int d^3p \left(a\, u(p,s)\, e^{-ipx} + b\, v(p,s)\, e^{ipx}\right) = \underline{\sum_s \int d^3p\; a\, u(p,s)\, e^{-ipx}}$$

og tilsvarende for P_-

For upolarisert spredning midles over innkommende og summeres over utgående polarisasjons retning:

$$\frac{1}{2}\sum_{\pm s_i}\sum_{\pm s_f} |\bar{u}_f \gamma^0 u_i|^2 = \frac{1}{2}\sum_{\pm s_i}\sum_{\pm s_f} \bar{u}_i \gamma^0 u_f\, \bar{u}_f \gamma^0 u_i$$

Utvider summen til å ta med både pos. og neg. energi ved å innføre proj. op.

$$= \frac{1}{2}\sum_{\substack{\pm s_i\\ \pm E_i}}\sum_{\substack{\pm s_f\\ \pm E_f}} \bar{u}_i \gamma^0 P_+ u_f\, \bar{u}_f \gamma^0 P_+ u_i$$

Fullstendighetsrelasjonen $\sum_{\substack{\pm s_f\\ \pm E_f}} u_f \bar{u}_f = 1$ gir så

$$= \frac{1}{2}\, Sp\left(\gamma^0 P_+ \gamma^0 P_+\right)$$

Gir her

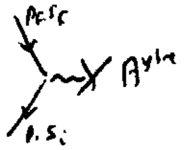
$$= \frac{1}{2}\, Sp\left(\gamma^0 \frac{m + \not{p}_f}{2m} \gamma^0 \frac{m + \not{p}_i}{2m}\right) = \frac{1}{2}\frac{1}{4m^2}\left(m^2\, Sp(\gamma^0\gamma^0) + p_{f\mu} p_{i\nu}\, Sp(\gamma^0\gamma^\mu\gamma^0\gamma^\nu)\right)$$

$$= \frac{1}{8m^2}\left(m^2 4 + 4\left(p_{f0} p_{i0} - p_{f\mu} p_i^{\mu} + p_{f0} p_{i0}\right)\right) = \frac{1}{2m^2}\left(m^2 + 2E_f E_i - (E_f E_i - \vec{p}_f \vec{p}_i)\right)$$

$$= \frac{1}{2m^2}\left(E^2 + m^2 + |\vec{p}|^2 \cos\vartheta\right) = \frac{E^2 - |\vec{p}|^2 \sin^2 \vartheta/2}{m^2}$$ da elastisk $E_f = E_i$, $|\vec{p}_f| = |\vec{p}_i|$, $m^2 = E^2 - (\vec{p})^2$

Tverrsnitt for upol. spredning

$$\left(\frac{d\sigma}{d\Omega}\right)_{upol} = 4\left(\frac{Ze^2}{4\pi}\right)^2 \frac{E^2 - |\vec{p}|^2 \sin^2 \vartheta/2}{\left(2|\vec{p}|\sin \vartheta/2\right)^4} \qquad |\vec{q}| = 2|\vec{p}|\sin\vartheta/2$$

c)  Vekselvirkningspotensial for elektron $V_e = -\frac{Ze^2}{4\pi}\frac{1}{r}$

for positron $V_p = \frac{Ze^2}{4\pi}\frac{1}{r}$

Gir $S^{(1)}_{pfi} = -i\frac{Ze^2}{(2\pi)^2}\frac{m}{\sqrt{E_f E_i}}\frac{\bar{v}_f\gamma^0 v_i}{|\vec{p}_f-\vec{p}_i|^2}\delta(E_f-E_i)$

som for elektronet bortsett fra faktoren -1

og u erstattet med v

Gir $\frac{d\sigma_p^{(1)}}{d\Omega} = 4\left(-\frac{Ze^2}{4\pi}\right)^2\frac{m}{|\vec{q}|^4}\left|\bar{v}_f\gamma^0 v_i\right|^2$ for polariserte positroner

For upolariserte spredning fås til 1. orden

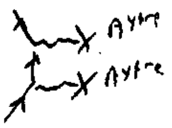
$$\underline{\left(\frac{d\sigma_p^{(1)}}{d\Omega}\right)_{upol} = \left(\frac{d\sigma_e^{(1)}}{d\Omega}\right)_{upol}}$$

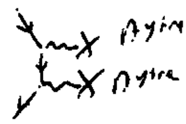
da $\frac{1}{2}\sum_{\pm s_i}\sum_{\pm s_f}\left|\bar{v}_f\gamma^0 v_i\right|^2 = \frac{1}{2}Sp\left(\gamma^0 P_-\gamma^0 P_-\right)$

$$= \frac{1}{2}Sp\left(\gamma^0\frac{m-\not{p}_f}{2m}\gamma^0\frac{m-\not{p}_i}{2m}\right) = \frac{1}{8m^2}\left(m^2 Sp(\gamma^0\gamma^0) + (-p_{f\mu})(-p_{i\nu})Sp(\gamma^0\gamma^\mu\gamma^0\gamma^\nu)\right)$$

Det samme som for elektronspredningen.

d) For elektronspredning . For positronspredning





Vekselvirkningspotensialet kommer inn kvadratisk slik at ~~[illegible]~~

$S^{(2)}_{efi} \sim \left(\frac{-Ze^2}{4\pi}\right)^2$ og $S^{(2)}_{pfi} \sim \left(\frac{Ze^2}{4\pi}\right)^2$ begge får samme fortegn.

mens $S^{(1)}_{efi}$ og $S^{(1)}_{pfi}$ hadde motsatt fortegn.

Dermed vil kryssleddene i tverrmiltene til 2. orden få motsatte fortegn ~~ved~~ elektron- og ved positron-spredning

$$\frac{d\sigma_e^{(2)}}{d\Omega} \sim \left|K^{(1)}_{efi} + K^{(2)}_{efi}\right|^2 \sim \left|K^{(1)}_{efi}\right|^2 + 2\,\mathrm{Re}\,K^{(1)}_{efi}K^{(2)}_{efi} + \left|K^{(2)}_{efi}\right|^2$$

$$\frac{d\sigma_p^{(2)}}{d\Omega} \sim \left|K^{(1)}_{pfi} + K^{(2)}_{pfi}\right|^2 \sim \left|K^{(1)}_{pfi}\right|^2 + 2\,\mathrm{Re}\,K^{(1)}_{pfi}K^{(2)}_{pfi} + \left|K^{(2)}_{pfi}\right|^2$$

$$\underline{\frac{\mathrm{Re}\,K^{(1)}_{efi}K^{(2)}_{efi}}{\mathrm{Re}\,K^{(1)}_{pfi}K^{(2)}_{pfi}} < 0} \qquad \underline{\underline{\frac{d\sigma_e^{(2)}}{d\Omega} \neq \frac{d\sigma_p^{(2)}}{d\Omega}}}$$