Løsningsforslag til vårekssamen 1994 i fag 74327 - Relativistisk kvantemekanikk

Oppgave 1

a)

Vi bruker Euler - Lagrange ligningene:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = 0 \Rightarrow \underline{\partial^\mu \partial_\mu \phi^* + m^2 \phi^* = 0}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi^*_{,\mu}} = 0 \rightarrow \underline{\partial^\mu \partial_\mu \phi + m^2 \phi = 0}$$

b)

$$\left.\begin{aligned} \phi' &= e^{ie\epsilon}\phi \\ \phi'^* &= e^{-ie\epsilon}\phi^* \end{aligned}\right\} \underline{\mathcal{L}' =} \partial^\mu \phi'^* \partial_\mu \phi' - m \phi'^* \phi'$$

$$= e^{-ie\epsilon} \partial^\mu \phi^* e^{ie\epsilon} \partial_\mu \phi - m^2 e^{-ie\epsilon} \phi^* e^{ie\epsilon} \phi$$

$$= \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi$$

$\mathcal{L}$ er invariant

$$\underline{= \mathcal{L}}$$

Infinitesimal transformasjon:

$$\phi \to e^{ie\epsilon}\phi \sim \phi + \underbrace{ie\epsilon\phi}_{=\delta\phi}$$

$$\phi^* \to e^{-ie\epsilon}\phi^* \sim \phi^* - \underbrace{ie\epsilon\phi^*}_{\delta\phi^*}$$

Nötherstrømmen:

$$\underline{j^\mu =} \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial \phi^*_{,\mu}}\,\delta\phi^*$$

$$= (\partial^\mu \phi^*)\,\delta\phi + (\partial^\mu\phi)\,\delta\phi^*$$

$$\underline{= ie\epsilon\,\{(\partial^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi)\}}$$

Nöther ladningen er:

$$\underline{Q =} \int d^3\vec{x}\, j^0 = \underline{ie\epsilon \int d^3\vec{x}\,\left((\partial^0\phi^*)\phi - \phi^*(\partial^0\phi)\right)}$$

Siden $j^\mu \to -j^\mu$ når $\begin{cases}\phi \to \phi^* \\ \phi^* \to \phi\end{cases}$.

Altså er det naturlig å assosiere ϕ og ϕ^* med motsatt ladde partikler.

c) Nå er

$$\mathcal{L} = (\partial^\mu \phi^* + ieA^\mu \phi^*)(\partial_\mu \phi - ieA_\mu \phi) - m^2 \phi^* \phi$$

Vi lar

$$\left.\begin{array}{l} \phi \to e^{ie\epsilon(x)} \phi \\ \phi^* \to e^{-ie\epsilon(x)} \phi^* \end{array}\right\}$$

$$\underline{\mathcal{L}'} = (\partial^\mu \phi'^* + ieA'^\mu \phi'^*)(\partial_\mu \phi' - ieA'_\mu \phi') - m^2 \phi'^* \phi'$$

$$= (e^{-ie\epsilon(x)} \partial^\mu \phi^* - ie\, e^{-ie\epsilon(x)} \phi^* \partial^\mu \epsilon + ieA'^\mu e^{-ie\epsilon(x)} \phi^*)$$

$$(e^{ie\epsilon(x)} \partial_\mu \phi + ie\, e^{ie\epsilon(x)} \phi\, \partial_\mu \epsilon - ieA' \phi)$$

$$- m^2 \phi^* \phi$$

$$= (\partial^\mu \phi^* + ie(A'^\mu - \partial^\mu \epsilon)\phi^*)$$

$$(\partial_\mu \phi - ie(A'_\mu - \partial_\mu \epsilon)\phi) - m^2 \phi^* \phi$$

$$= (\partial^\mu \phi^* + ieA^\mu \phi^*)(\partial_\mu \phi - ieA_\mu \phi) - m^2 \phi^* \phi$$

$$\underline{= \mathcal{L}}$$

hvis $\underline{A'^\mu = A^\mu + \partial^\mu \epsilon}$

d) Infinitesimal transformasjon:

$$\begin{cases} \delta\phi = ie\,\epsilon(x)\,\phi \\ \delta\phi^* = -ie\,\epsilon(x)\,\phi^* \\ \delta A_\mu = \partial_\mu \epsilon \end{cases}$$

$\mathcal{L}$ avhenger ikke av $A_{\mu,\nu}$

Nøtherstrømmen:

$$\underline{j^\mu} = \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial \phi^*_{,\mu}}\,\delta\phi^* + \underbrace{\cancel{\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}}}}_{=0}\,\delta A_\nu$$

$$= ie\epsilon\,\{ (\partial^\mu\phi^* + ieA^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi - ieA^\mu\phi) \}$$

$$\partial_\mu j^\mu = ie\,(\partial_\mu\epsilon)\{ (\partial^\mu\phi^* + ieA^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi - ieA^\mu\phi) \}$$

$$+ ie\epsilon\,\underbrace{\partial_\mu\{ (\partial^\mu\phi^* + ieA^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi - ieA^\mu\phi) \}}$$

Tar for oss denne på neste side

Bevegelsesligningene (fra Euler-Lagrange ligningene):

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_\mu} = 0 \Rightarrow (\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu)\phi^* + m^2\phi^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi^*_{;\mu}} = 0 \Rightarrow (\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu)\phi + m^2\phi = 0$$

Skriver disse ut:

$$(\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu)\phi^* + m^2\phi^* =$$

$$\partial_\mu \partial^\mu \phi^* + 2ieA_\mu \partial^\mu \phi^* + ie(\partial_\mu A^\mu)\phi^*$$
$$- e^2 A_\mu A^\mu \phi^* + m^2\phi^* = 0$$

$$(\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu)\phi + m^2\phi =$$

$$\partial_\mu \partial^\mu \phi - 2ieA_\mu \partial^\mu \phi + ie(\partial_\mu A^\mu)\phi$$
$$- e^2 A_\mu A^\mu \phi + m^2\phi = 0$$

Vi vender nå tilbake til side 4:

$$\partial_\mu \{ (\partial^\mu \phi^* + ieA^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi - ieA^\mu \phi) \}$$

$$= \partial_\mu \{ (\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu\phi) + 2ieA^\mu \phi^*\phi \}$$

$$= (\partial_\mu \partial^\mu \phi^*)\phi + (\partial^\mu\phi^*)(\partial_\mu\phi) - (\partial_\mu\phi^*)(\partial^\mu\phi)$$
$$- \phi^*(\partial_\mu\partial^\mu\phi) + 2ie\,(\partial_\mu A^\mu)\phi^*\phi$$
$$+ 2ieA^\mu(\partial_\mu\phi^*)\phi + 2ieA^\mu\phi^*\partial_\mu\phi$$

$$= \phi\{ \partial_\mu\partial^\mu\phi^* + 2ieA^\mu\partial_\mu\phi^* + ie\,(\partial_\mu A^\mu)\phi^* \}$$
$$- \phi^*\{ \partial_\mu\partial^\mu\phi - 2ieA^\mu\partial_\mu\phi - ie\,(\partial_\mu A^\mu)\phi \}$$

Bruker bevegelseslikningene fra side 5:

$$= \phi\{ e^2 A_\mu A^\mu \phi^* - m^2\phi^* \} - \phi^*\{ e^2 A_\mu A^\mu\phi - m^2\phi \}$$
$$\underline{= 0}$$

Altså:

$$\underline{\partial_\mu J^\mu =} \; \underline{ie\,(\partial_\mu \epsilon)\{ (\partial^\mu\phi^* + ieA^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi - ieA^\mu\phi) \}}$$

Denne er null hvis $\partial_\mu \epsilon = 0 \Longrightarrow$

<u>ϵ er en konstant.</u>

Oppgave 2

a) $\mathcal{K}_{ret}(x_2, x_1)$ fører både partikler og antipartikler framover i tid.

$\mathcal{K}_{adv}(x_2, x_1)$ fører både partikler og antipartikler bakover i tid.

$iS_F(x_2, x_1)$ fører partikler framover i tid og antipartikler bakover i tid.

b)

$$\underline{(i\not{\partial}_2 - m)\, iS_F(x_2, x_1)} = (i\gamma^0\partial^0 - i\vec{\gamma}\cdot\vec{\nabla} - m)\, iS_F(x_2, x_1)$$

$$= i\gamma^0 \int \frac{d^3\vec{p}}{2p^0(2\pi)^3} \Big\{ \delta(t_2 - t_1)(\not{p} + m)\, e^{-ip(x_2 - x_1)}$$

$$- \delta(t_1 - t_2)(\not{p} - m)\, e^{+ip(x_2 - x_1)} \Big\}$$

$$+ \int \frac{d^3\vec{p}}{2p^0(2\pi)^3} \Big\{ \theta(t_2 - t_1)\underbrace{(\not{p} + m)(\not{p} - m)}_{=0} e^{-ip(x_2 - x_1)}$$

$$- \theta(t_1 - t_2)\underbrace{(\not{p} - m)(-\not{p} - m)}_{=0} e^{ip(x_2 - x_1)} \Big\}$$

$(\not{p} - m)(\not{p} + m) = (\not{p} + m)(\not{p} - m) = 0$

$$= i\gamma^0 \int \frac{d^3\vec{p}}{2p^0(2\pi)^3} \Big\{ \delta(t_2-t_1)(\not{p}+m)\, e^{i\vec{p}\cdot(\vec{x}_2-\vec{x}_1)}$$

$$- \delta(t_1-t_2)(\not{p}-m)\, e^{-i\vec{p}\cdot(\vec{x}-\vec{x}_1)} \Big\}$$

$$= i\gamma^0 \delta(t_1-t_2) \int \frac{d^3\vec{p}}{2p^0(2\pi)^3} \Big\{ (\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} + m)\, e^{i\vec{p}\cdot(\vec{x}_2-\vec{x}_1)}$$

$$- \underbrace{(\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} - m)\, e^{-i\vec{p}(\vec{x}-\vec{x}_1)}} \Big\}$$

forandrer integrasjonsvariabel $\vec{p} \to -\vec{p}$ her

$$= i\gamma^0 \delta(t_1-t_2) \int \frac{d^3\vec{p}}{2p^0(2\pi)^3} \big(2\gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} + m$$

$$+ \vec{\gamma}\cdot\vec{p} - m \big)\, e^{i\vec{p}(\vec{x}_2-\vec{x}_1)}$$

$$= i\gamma^0 \delta(t_1-t_2) \int \frac{d^3\vec{p}}{2p^0(2\pi)^3}\, 2\gamma^0 p^0\, e^{i\vec{p}(\vec{x}_2-\vec{x}_1)}$$

$$= i\,\delta(t_1-t_2) \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}_2-\vec{x}_1)} \underline{= i\,\delta^{(4)}(x_2-x_1)}$$

c) Vi multipliserer

$$(i\not{\partial} - e\not{A}(x) - m)\, S_A(x, x_1) = \delta(x-x_1)$$

med $S_F(x_2, x)$ på venstre.

Integrerer så over x:

$$S_F(x_2,x_1) = \int dx\, S_F(x_2,x)\,\delta(x-x_1)$$

derivasjon til høyre

$$= \int dx\, S_F(x_2,x)\,(i\overrightarrow{\not\partial} - e\not A(x) - m)\,S_A(x,x_1)$$

$$= \int dx\, S_F(x_2,x)\,(i\overrightarrow{\not\partial} - m)\,S_A(x,x_1)$$

$$- e\int dx\, S_F(x_2,x)\not A(x)\,S_A(x,x_1)$$

partiell integrasjon

$$= \int dx\, S_F(x_2,x_1)\,(-i\overleftarrow{\not\partial} - m)\,S_A(x,x_1)$$

$$- e\int dx\, S_F(x_2,x)\not A(x)\,S_A(x,x_1) = *$$

Nå bruker vi trikset:

$$S_F(x_2,x)(-i\not\partial - m) = \delta(x_2 - x)$$

$$* = \int dx\, \delta(x_2-x)\,S_A(x,x_1)$$

$$- e\int dx\, S_F(x_2,x)\not A(x)\,S_A(x,x_1)$$

$$= S_A(x_2,x_1) - e\int dx\, S_F(x_2,x)\not A(x)\,S_A(x,x_1)$$

Vi ommøblerer:

$$S_A(x_2,x_1) = S_F(x_2,x_1) + e\int dx\, S_F(x_2,x)\not{A}(x)\, S_A(x,x_1)$$

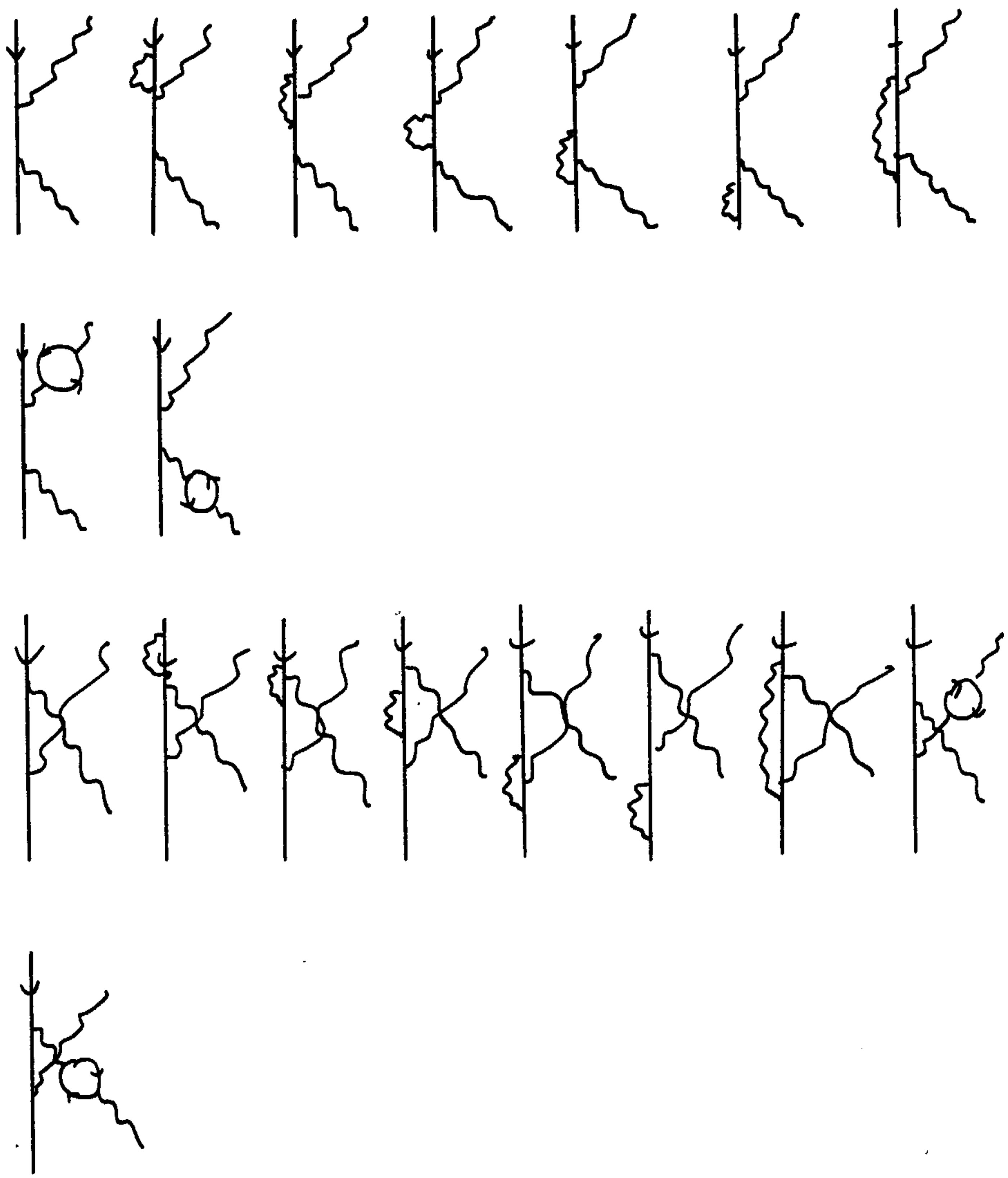
Vi genererer en perturbasjonsekspansjon gjennom å substituere S_A til høyre over med uttrykk av samme type.

$$\begin{aligned}
S_A(x_2,x_1) &= S_F(x_2,x_1) + e\int dx\, S_F(x_2,x)\not{A}(x)\, S_A(x,x_1) \\
&= S_F(x_2,x_1) + e\int dx\, S_F(x_2,x)\not{A}(x)\Big(S_F(x,x_1) \\
&\qquad + e\int dy\; S_F(x,y)\not{A}(y)\, S_A(y,x_1)\Big) \\
&= S_F(x_2,x_1) + e\int dx\, S_F(x_2,x)\not{A}(x)\; S_F(x,x_1) \\
&\qquad + e^2\int dx\, dy\;\; S_F(x_2,x)\not{A}(x)\, S_F(x,y)\not{A}(y)\, S_A(y,x_1)
\end{aligned}$$

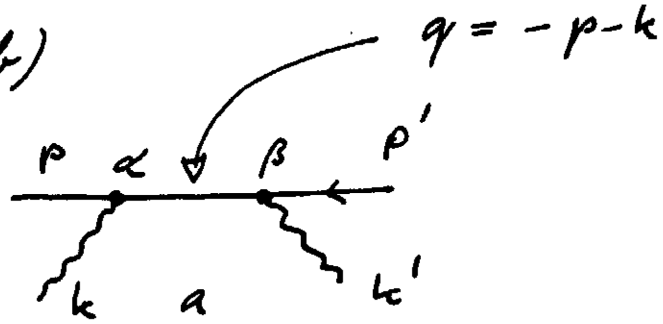
Nå kan vi substituere igjen og igjen og ....

Oppgave 3

a)

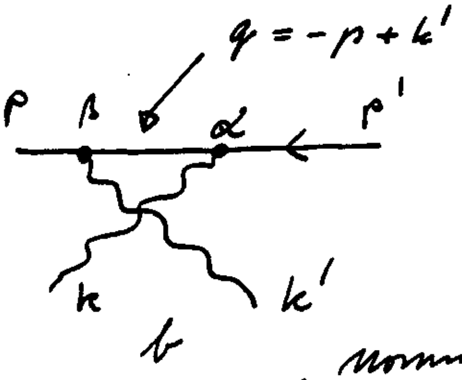


b)



$$\mathcal{M}_a = -(ie)^2 \bar{v}(p)\, \epsilon_\alpha(k)\, \gamma^\alpha \frac{i}{-\not{p}-\not{k}-m+i\epsilon} \gamma^\beta \epsilon_\beta(k')\, v(p')$$

$$= -(ie)^2 \bar{v}(p)\, \not{\epsilon}(k) \frac{i}{-\not{p}-\not{k}-m+i\epsilon} \not{\epsilon}(k')\, v(p')$$



normal ordering qui [illegible] [illegible].

$$\mathcal{M}_b = -(ie)^2 \bar{v}(p)\, \epsilon_\beta(k')\, \gamma^\beta \frac{i}{-\not{p}+\not{k}'-m+i\epsilon} \gamma^\alpha \epsilon_\alpha(k)\, v(p')$$

$$= -(ie)^2 \bar{v}(p)\, \not{\epsilon}(k') \frac{i}{-\not{p}+\not{k}'-m+i\epsilon} \not{\epsilon}(k)\, v(p')$$

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$$

$$= -(ie)^2 \bar{v}(p) \left\{ \not{\epsilon}(k) \frac{i}{-\not{p}-\not{k}-m+i\epsilon} \not{\epsilon}(k') + \not{\epsilon}(k') \frac{i}{-\not{p}+\not{k}'-m+i\epsilon} \not{\epsilon}(k) \right\} v(p)$$