

Kontinuitätsgleichungen i Relativistisk Kvantemekanik

Jag 74327 - Sommer 1994

Oppgave 1:

$$\begin{aligned}
 a) \quad \underline{i \frac{\partial}{\partial t} |\psi_I\rangle} &= i \frac{\partial}{\partial t} e^{iH_0 t} |\psi_S\rangle \\
 &= -H_0 e^{iH_0 t} |\psi_S\rangle + e^{iH_0 t} i \frac{\partial}{\partial t} |\psi_S\rangle \\
 &= -H_0 |\psi_I\rangle + e^{iH_0 t} H |\psi_S\rangle \\
 [H_0, V] &= 0 \rightarrow \\
 &= -H_0 |\psi_I\rangle + H e^{iH_0 t} |\psi_S\rangle \\
 &= \underline{\underline{V |\psi_I\rangle}}
 \end{aligned}$$

$$b) \quad \left. \begin{array}{l} \text{La } t_f = t_N \\ t_i = t_0 \end{array} \right\} \Delta t = \frac{t_N - t_0}{N} \quad \begin{array}{l} N \text{ er stor nok til} \\ \text{at } V \text{ er} \\ \text{konstant i} \\ \text{intervallet } \Delta t \end{array}$$

$$\begin{aligned}
 i \frac{\partial}{\partial t} |\psi_I\rangle &= i \frac{\partial}{\partial t} U(t, t_0) |\psi_I\rangle \\
 &= V U(t, t_0) |\psi_I\rangle \Rightarrow
 \end{aligned}$$

$$i \frac{\partial}{\partial t} U(t, t_0) = V U(t, t_0)$$

Indføre sammenligning:

$$U(t_i, t_i - \Delta t) = e^{-i \Delta t V(t_i)}$$

$$\approx (1 - i \Delta t V(t_i))$$

$$U(t_N, t_0) = U(t_N, t_{N-1}) U(t_{N-1}, t_{N-2}) \dots U(t_1, t_0)$$

$$= (1 - i \Delta t V(t_N)) (1 - i \Delta t V(t_{N-1})) \dots$$

$$= \mathcal{T} \{ (1 - i \Delta t V(t_N)) (1 - i \Delta t V(t_{N-1})) \dots \}$$

↑
Tidsordningsoperator - når kan vi bytte
om på operatorerne som vi vil

$$= \mathcal{T} \left\{ 1 + (-i) \Delta t \sum_{i=1}^N V(t_i) + \frac{(-i)^2}{2!} \Delta t \left(\sum_{i=1}^N V(t_i) \right)^2 \right.$$

$$+ \dots = \mathcal{T} e^{-i \Delta t \sum_{i=1}^N V(t_i)}$$

$\Rightarrow$

$$U(t_f, t_i) = \mathcal{T} e^{-i \int_{t_i}^{t_f} dt V(t)}$$

c)

$$V(t) = -f(t)x(t)$$

$$\int_{t_i}^{t_f} V(t) dt = - \int_{t_i}^{t_f} f(t)x(t) dt$$

$$= - \sqrt{\frac{2}{m\omega}} \left\{ a \int_{t_i}^{t_f} dt f(t) e^{-i\omega t} + a^\dagger \int_{t_i}^{t_f} dt f(t) e^{i\omega t} \right\}$$

$$= - \sqrt{\frac{2}{m\omega}} (a \tilde{f}(\omega)^* + a^\dagger \tilde{f}(\omega))$$

$$\begin{aligned} U(t_f, t_i) &= \mathcal{T} e^{-i \int_{t_i}^{t_f} V(t) dt} = \mathcal{T} e^{i \sqrt{\frac{2}{m\omega}} (a \tilde{f}(\omega)^* + a^\dagger \tilde{f}(\omega))} \\ &= e^{i \sqrt{\frac{2}{m\omega}} (a \tilde{f}(\omega)^* + a^\dagger \tilde{f}(\omega))} \end{aligned}$$

↑
Siden a og $a^\dagger$ er tidsuafhængige.

d)

Nåen Baker-Hausdorff formel til 2. orden:

$$\underline{e^{A+B} = 1 + (A+B) + \frac{1}{2}(A^2 + AB + BA + B^2) + \dots}$$

$$\underline{e^A e^B e^{-\frac{1}{2}[A,B]}} = (1 + A + \frac{1}{2}A^2 + \dots)(1 + B + \frac{1}{2}B^2 + \dots)(1 - \frac{1}{2}[A,B] + \dots)$$

$$= 1 + (A+B) + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 - \frac{1}{2}AB + \frac{1}{2}BA$$

$$\underline{= 1 + (A+B) + \frac{1}{2}(A^2 + AB + BA + B^2) + \dots}$$

et er gjort i 3. runde at man får
brukt at $[A, [A, B]] = [B, [A, B]] = 0$.

her er en full utledning:

Vi får brukt at

$$\underline{[A^n, B] = n A^{n-1} [A, B]}$$

som gjelder hvis $[A, [A, B]] = [B, [A, B]] = 0$

Da kan vi vite at

$$\begin{aligned}\underline{e^A B e^{-A}} &= \sum_{n=0}^{\infty} \frac{1}{n!} A^n B e^{-A} \\&= \sum_{n=0}^{\infty} \frac{1}{n!} (B A^n + [A^n, B]) e^{-A} \\&= B \sum_{n=0}^{\infty} \frac{1}{n!} A^n e^{-A} + \sum_{n=1}^{\infty} \frac{1}{n!} [A^n, B] e^{-A} \\&= B + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} A^{n-1} [A, B] e^{-A} \\&= B + [A, B] \sum_{n=0}^{\infty} \frac{1}{n!} A^n e^{-A} \\&= \underline{B + [A, B]}\end{aligned}$$

(5)

Definiți să:

$$f(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$$

$$\frac{d}{d\lambda} f(\lambda) = A e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$$

$$+ e^{\lambda A} B e^{\lambda B} e^{-\lambda(A+B)}$$

$$- e^{\lambda A} e^{\lambda B} (A+B) e^{-\lambda(A+B)}$$

$$= A f + e^{\lambda A} e^{\lambda B} B e^{-\lambda(A+B)}$$

$$- e^{\lambda A} e^{\lambda B} A e^{-\lambda(A+B)}$$

$$- e^{\lambda A} e^{\lambda B} B e^{-\lambda(A+B)}$$

$$= A f - e^{\lambda A} (A + \lambda[B, A]) e^{\lambda B} e^{-\lambda(A+B)}$$

$$e^A B e^{-A} = B + [A, B]$$

$$= \lambda [A, B] f(\lambda)$$

Integrăm dinu oz lui:

$$f(\lambda) = e^{\frac{1}{2} \lambda^2 [A, B]}$$

⑥

Alt π^0 :

$$e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)} = e^{\frac{1}{2}\lambda^2[A,B]}$$

$$\Rightarrow \boxed{e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}}$$

e)

$$\begin{aligned} \underline{U(t_f, t_i)} &= e^{i(a^+ \tilde{f}(\omega) + a^+ \tilde{f}^*(\omega))} \\ &= e^{i a^+ \tilde{f}(\omega)} e^{i a \tilde{f}^*(\omega)} e^{-\frac{1}{2}[i a^+ \tilde{f}(\omega), i a \tilde{f}^*(\omega)]} \\ &= e^{i a^+ \tilde{f}(\omega)} e^{i a \tilde{f}^*(\omega)} e^{-\frac{1}{2}|\tilde{f}(\omega)|^2} \\ &\quad ([a^+, a] = 1) \end{aligned}$$

f)

$$\begin{aligned} \underline{U(t_f, t_i)|0\rangle} &= e^{i f a^+} e^{i f^* a} |0\rangle e^{-\frac{1}{2}|f|^2} \\ &= e^{i f a^+} |0\rangle e^{-\frac{1}{2}|f|^2} \\ &= \sum_{k=0}^{\infty} \frac{i^k}{k!} f^k a^{+k} |0\rangle e^{-\frac{1}{2}|f|^2} \end{aligned}$$

$$|n\rangle = \frac{1}{\sqrt{n!}} a^{+n} |0\rangle$$

⑦

$$A_n = \langle n | U(t_i, t_i) | 0 \rangle$$

$$= e^{-\frac{1}{2}|f|^2} \sum_{k=0}^{\infty} \frac{i^k}{k!} f^k \langle n | k \rangle$$

$$= \frac{e^{-\frac{1}{2}|f|^2}}{\sqrt{n!}} (if)^n$$

$$\underline{\underline{P_n = (A_n)^2 = \frac{|f|^{2n} e^{-|f|^2}}{n!}}}$$

(Poisson probability)

Mittelwert berechnen:

$$\underline{\underline{\langle n \rangle = \sum_{n=0}^{\infty} n P_n = e^{-|f|^2} \sum_{n=0}^{\infty} \frac{|f|^{2n}}{(n-1)!}}}$$

$$= |f|^2 e^{-|f|^2} \sum_{n=0}^{\infty} \frac{|f|^{2n}}{n!}$$

$$\underline{\underline{= |f|^2}}$$

g) $\hat{H}_0$ u

$$U(t_i, t_i) = e^{i \sum_{k,\alpha} \sqrt{\frac{1}{2\omega}} (a_{k\alpha}^\dagger j_{k\alpha} + a_{k\alpha} j_{k\alpha}^*)}$$

hier

$$j_{k\alpha} = \sqrt{\frac{1}{V}} \int d^3x \epsilon_\alpha \cdot \vec{j}(x) e^{i\vec{k} \cdot \vec{x}}$$

$$\omega = k^0 = |\vec{k}|$$

(8)

Brucke Baker-Flamendorf theorem of art

$$[a_{k,\alpha}^\dagger, a_{p,\beta}] = \delta_{kp} \delta_{\alpha\beta}$$

$$U(t_f, t_i) = e^{i \sum_{k,\alpha} \frac{1}{\sqrt{2\omega}} a_{k,\alpha}^\dagger f_k(k)} e^{i \sum_{k,\alpha} \frac{1}{\sqrt{2\omega}} a_{k,\alpha} f_k^*(k)} e^{-\frac{1}{2} \sum_{k,\alpha} \frac{|f_k(k)|^2}{2\omega}}$$

Analogi med punkt f) har vi så at

$$\langle 0 | U(t_f, t_i) | 0 \rangle = e^{-\frac{1}{2} \sum_{k,\alpha} \frac{|f_k(k)|^2}{2\omega}}$$

$$\underline{P_0 = \langle 0 | U(t_f, t_i) | 0 \rangle}^2 = e^{-\sum_{k,\alpha} \frac{|f_k(k)|^2}{2\omega}}$$

$$= \prod_{k,\alpha} e^{-\frac{|f_k(k)|^2}{2\omega}}$$

$$\langle 1_{k,\alpha} | U(t_f, t_i) | 0 \rangle = \langle 0 | a_{k,\alpha} U(t_f, t_i) | 0 \rangle$$

$$= \frac{i}{\sqrt{2\omega}} f_k(k) e^{-\frac{1}{2} \sum_{k',\alpha'} \frac{|f_{k'}(k')|^2}{2\omega}}$$

$$\underline{P_{1_{k,\alpha}}} = \langle 1_{k,\alpha} | U(t_f, t_i) | 0 \rangle^2 = \frac{|f_k(k)|^2}{2\omega} \prod_{k',\alpha'} e^{-\frac{|f_{k'}(k')|^2}{2\omega}}$$

(9)

Relativ sannsynlighet for emisjon av et foton:

$$\underline{p_{k\alpha} = \frac{P_{k\alpha}}{P_0} = \frac{|\dot{\gamma}_{\alpha}(k)|^2}{2\omega}}$$

Relativ sannsynlighet for emisjon av to fotoner med energi $k_1 \neq k_2$; $\alpha_1 \neq \alpha_2$:

$$p_{k_1, \alpha_1} p_{k_2, \alpha_2}$$

Hvis $k_1 = k_2$, $\alpha_1 = \alpha_2$:

$$\frac{1}{2} p_{k\alpha}^2$$

↑
like partikler.

Absolutt sannsynlighet:

$$\boxed{p_{k_1, \alpha_1} p_{k_2, \alpha_2} P_0} \quad \text{og} \quad \boxed{\frac{1}{2} p_{k\alpha}^2 P_0}$$

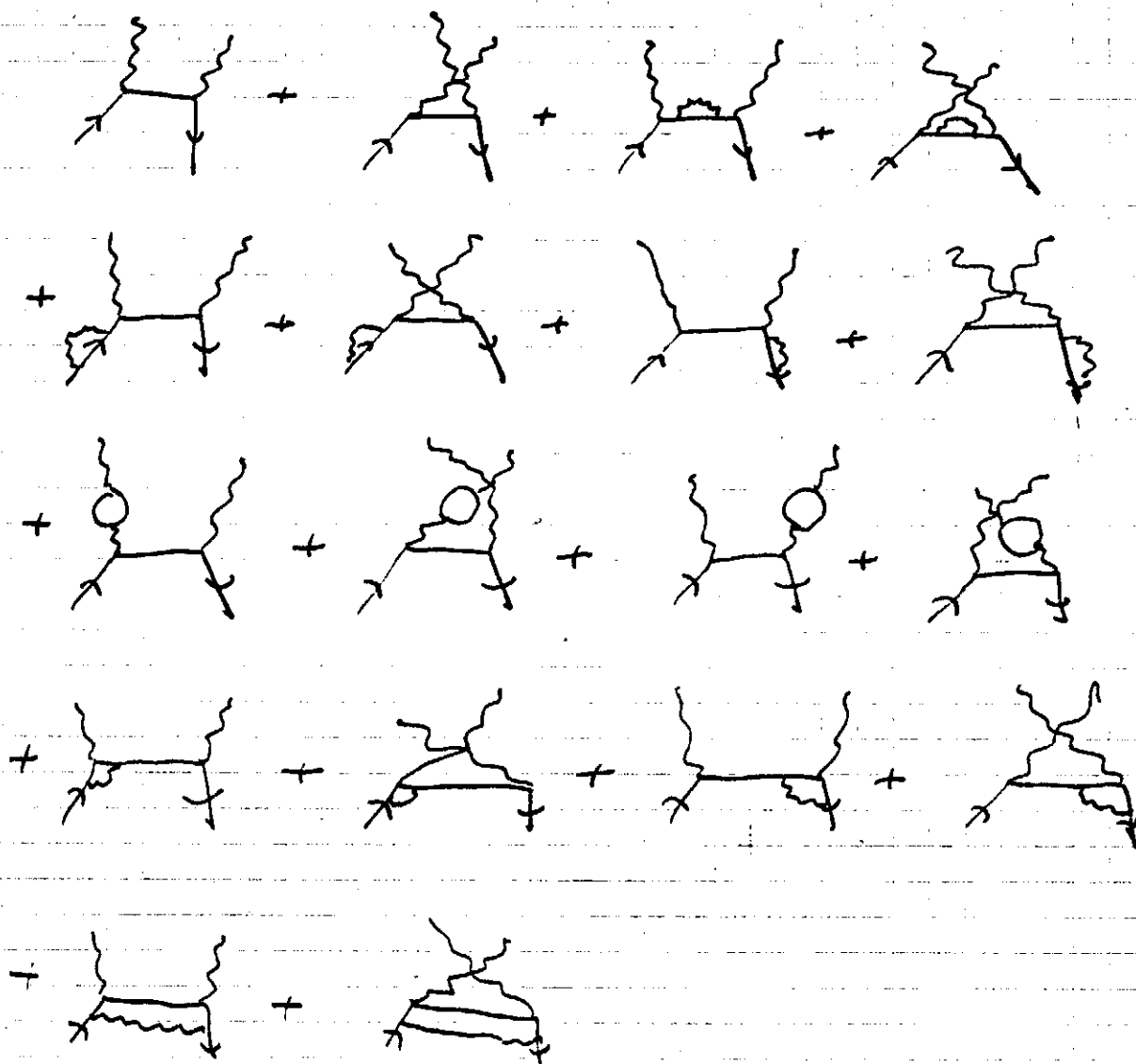
Midlere antall fotoner:

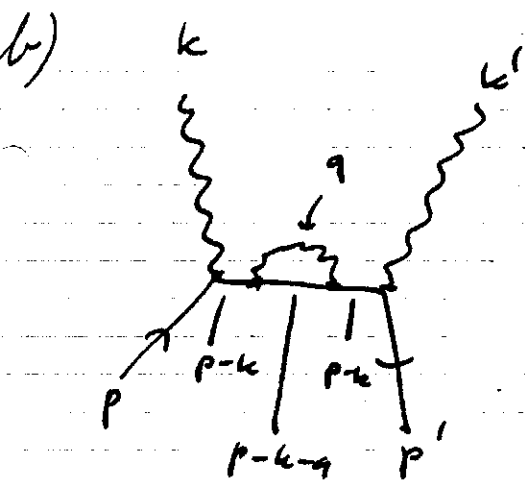
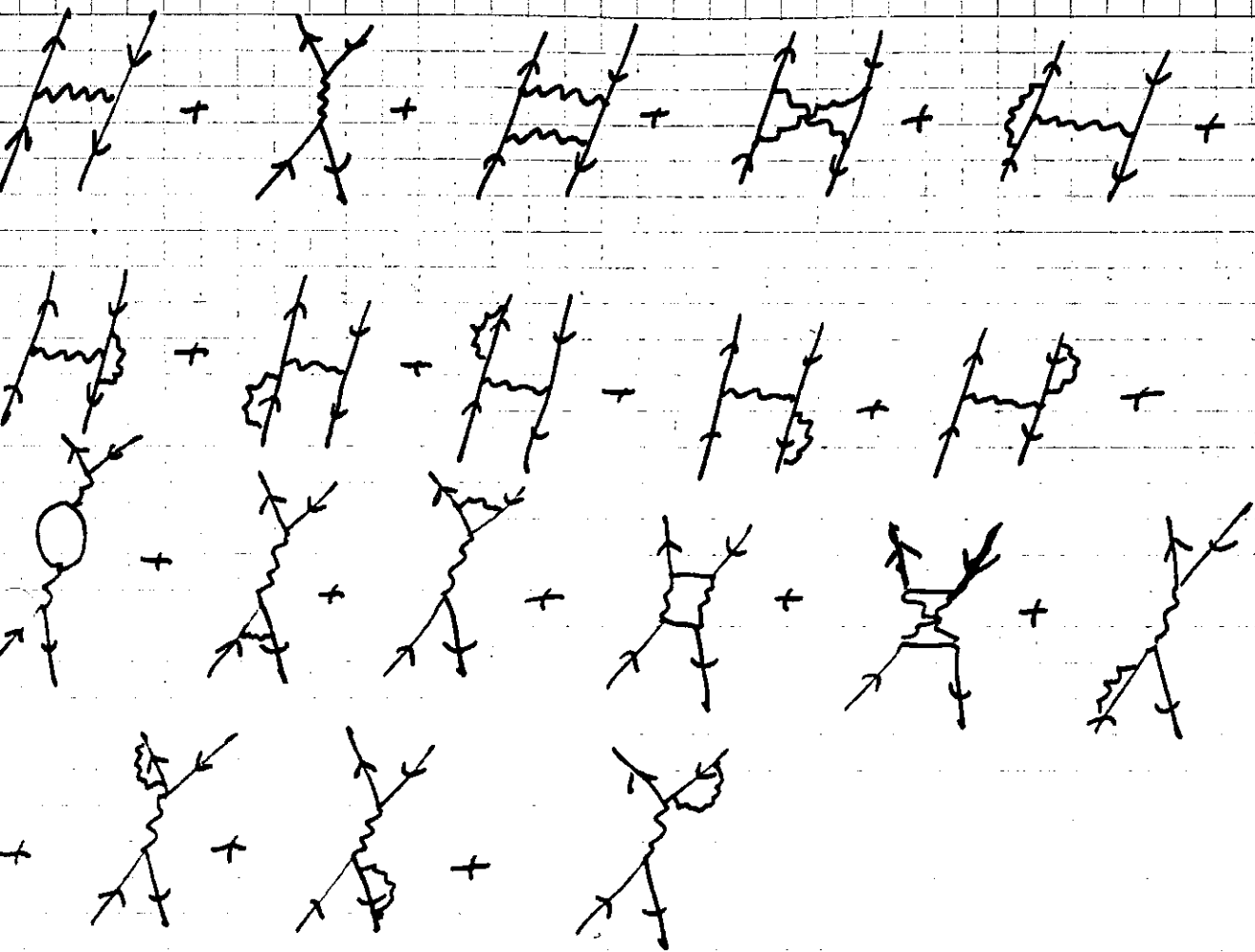
$$\text{for } f) : \langle n \rangle = |f|^2$$

$$\underline{\langle N_{k\alpha} \rangle = \frac{|\dot{\gamma}_{\alpha}(k)|^2}{2\omega}}$$

10) Сурдари 2

a) $e^+e^- \rightarrow \gamma\gamma$

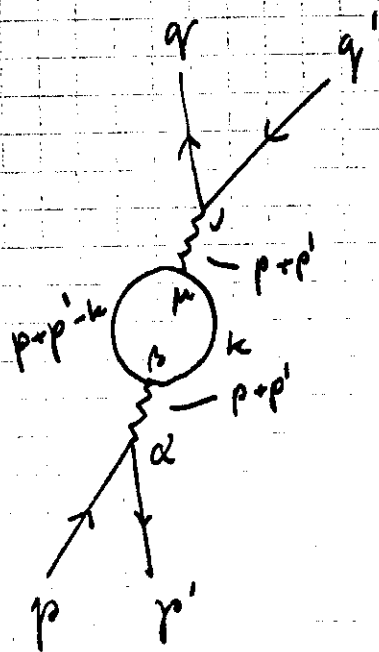




$$p + p' = q + q'$$

$$dM = \bar{u}(p') ie \not{\epsilon}(k') iS_F(p-k) \frac{1}{(2\pi)^4} \int d^4q$$

$$iD_{F\beta\alpha}(q) \not{\epsilon} \sigma^p iS_F(p-k-q) ie \sigma^\alpha iS_F(p-k) ie \not{\epsilon}(k) u(p)$$



$$p+p' = q+q'$$

$$\mathcal{M} = \bar{u}(q) i e \gamma^\nu v(q') i D_{F\nu\mu}(p+p') i D_{F\mu\alpha}(p+p') \bar{v}(p') i e \gamma^\alpha u(p)$$

$$\frac{(-i)}{(2\pi)^4} \int d^4k \text{tr} (i S_F(k) i e \gamma^\beta i S_F(p+p'-k) i e \gamma^\mu)$$

$$i D_{F\beta\alpha}(p+p') \bar{v}(p') i e \gamma^\alpha u(p)$$

- c) Prosessen av typen $e^- \rightarrow e^- \gamma$ etc. har amplitude null hvis prosessen skal oppfylle impuls-konservering samtidig som at fotonet skal være massløst.

d) Vi spesifiserte nøyen dynamiske egenskaper for den klassiske strømmen i 1g, og har derfor et bort fra impuls-konservering.