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Løsningsforslag til vizekannen 1995 i fag 74327:
Relativistisk mekanikk

Oppgave 1

a) Skriv ut (2) på komponentform:

$$\frac{\partial}{\partial t} E_k = \epsilon_{ijk} \frac{\partial}{\partial x_i} B_j$$

Altså:

$$i \frac{\partial}{\partial t} E_k = -1 \cdot i \epsilon_{ijk} \frac{\partial}{\partial x_i} i B_j \\ = -i (-i) \epsilon_{ikj} \frac{\partial}{\partial x_i} i B_j \quad (A)$$

↑
forbyrskifte p.g.a.
permutering av
indeksene.

Braker vi (7) nå, kan vi skrive (A)
som:

$$i \frac{\partial}{\partial t} E_k = -i (S_i)_{ki} \frac{\partial}{\partial x_i} i B_j \quad (B)$$

På vektor form blir dette (5).

Omformingen av (4) til (6) er helt analog.

b)

$$\underline{[S_i, S_j]_{mn}} = - \{ \epsilon_{imk} \epsilon_{jkn} - \epsilon_{jmk} \epsilon_{ikn} \}$$

$$= \epsilon_{kim} \epsilon_{kjn} - \epsilon_{kim} \epsilon_{kin}$$

$$= \delta_{ij} \delta_{mn} - \delta_{in} \delta_{jm} - \cancel{\delta_{ji} \delta_{mn}} + \delta_{jn} \delta_{mi}$$

$$= \delta_{jn} \delta_{im} - \delta_{jm} \delta_{in} = \epsilon_{kji} \epsilon_{knm}$$

$$= -\epsilon_{ijk} \epsilon_{knm} = \epsilon_{ijk} \epsilon_{knm}$$

$$= i \epsilon_{ijk} (-i) \epsilon_{knm}$$

$$= \underline{i \epsilon_{ijk} (S_k)_{mn}} \quad (c)$$

c)

Skriver vi ut (11) på 2×2 form, får vi

$$i \frac{\partial}{\partial t} \begin{Bmatrix} \vec{E} \\ i\vec{0} \end{Bmatrix} = -i \begin{Bmatrix} 0 & \vec{S} \\ \vec{S} & 0 \end{Bmatrix} \begin{Bmatrix} \vec{E} \\ i\vec{0} \end{Bmatrix} \quad (d)$$

som på 2-komponent form ges (5) or (6).

d) Vi "vadaren" (11):

$$\begin{aligned}
 -\frac{\partial^2}{\partial t^2} \psi &= -i\vec{\alpha} \cdot \vec{\nabla} i\frac{\partial}{\partial t} \psi = -(\vec{\alpha} \cdot \vec{\nabla})(\vec{\alpha} \cdot \vec{\nabla}) \psi \\
 &= -\alpha_i \alpha_j \frac{\partial^2}{\partial x_i \partial x_j} \psi
 \end{aligned}$$

Aden $\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}$

$$\begin{aligned}
 &= -\frac{1}{2} (\alpha_i \alpha_j + \alpha_j \alpha_i) \frac{\partial^2}{\partial x_i \partial x_j} \psi \\
 &= -\frac{1}{2} \{\alpha_i, \alpha_j\} \frac{\partial^2}{\partial x_i \partial x_j} \psi
 \end{aligned}$$

Altså:
$$\frac{\partial^2}{\partial t^2} \psi = \frac{1}{2} \{\alpha_i, \alpha_j\} \frac{\partial^2}{\partial x_i \partial x_j} \psi \quad (E)$$

Maxwells ligninger:

$$\vec{\nabla} \times \left| \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} \right.$$

$$\frac{\partial}{\partial t} \vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}$$

$\nabla \cdot \vec{B} = 0$ p.g.a. (3)

Bruker (4)

$$\begin{aligned}
 &\downarrow \\
 -\frac{\partial^2}{\partial t^2} \vec{B} &= -\nabla^2 \vec{B} \rightarrow \underline{\underline{\frac{\partial^2}{\partial t^2} i\vec{B} = \nabla^2 i\vec{B} \quad (F)}}
 \end{aligned}$$

På samme måde:

$$\vec{\nabla} \times \left| \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \right.$$

$$\frac{\partial}{\partial t} \nabla \times \vec{B} = -\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} (\vec{\nabla} \cdot \vec{E}) + \nabla^2 \vec{E}$$

= 0 p.g.g.

ligning (2)

$$\frac{\partial^2}{\partial t^2} \vec{E} = \nabla^2 \vec{E} \quad (6)$$

Skriver (F) og (G) ved hjælp af ψ , defineret i (10), kan man:

$$\frac{\partial^2}{\partial t^2} \psi = \nabla^2 \psi \quad (H)$$

Sammeltiden man (E) og (H) får man at

$$\underline{\underline{\{ \alpha_i, \alpha_j \} = 2 \delta_{ij}}} \quad (I)$$

c) En ligning af formen $i \partial^\mu \partial_\mu \psi = 0$

er kan kovariant hvis den står en Lorentz

transformering så det om $i \partial^\mu \partial'_\mu \psi' = 0$

δ -matricen er de samme som for transformering

f) Vi antager at (14) er kovariant. Da kan vi:

$$i \gamma^\mu \frac{\partial}{\partial x'^\mu} \psi'(x') = 0 \quad \rightarrow$$

$$i \gamma^\mu \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}} \frac{\partial}{\partial x^\nu} L(1) \psi(x) = 0$$

$$\uparrow$$

$$x^\mu = x^\mu(x') = \frac{\partial x^\mu}{\partial x'^\nu} x'^\nu = (\Lambda^{-1})^\mu_\nu x'^\nu$$

$$L(1)^{-1} \mid i \gamma^\mu (\Lambda^{-1})^\nu_\mu L(1) \frac{\partial}{\partial x^\nu} \psi(x) = 0$$

$$i \underbrace{L(1)^{-1} \gamma^\mu L(1) (\Lambda^{-1})^\nu_\mu}_{\text{hvis kovariant}} \frac{\partial}{\partial x^\nu} \psi(x) = 0$$

$$\text{hvis kovariant:} = \gamma^\nu$$

$$\rightarrow \underline{L(1)^{-1} \gamma^\mu L(1) (\Lambda^{-1})^\nu_\mu = \gamma^\nu}$$

Som er meningen (15).

g) (14) er ikke kovariant. Velg $\mu=0$ i (15).

Da kan vi ved brugen af (12):

$$L(1) \gamma^0 L(1)^{-1} = L(1) L(1)^{-1} = 1 = \underline{(\Lambda^{-1})^0_\nu \gamma^\nu}$$

Som er stridmodsigende.

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Oppgave 2

a)

This $C^{-1} = i\gamma^0\gamma^3$, si må vi ha at

$$(i\gamma^2\gamma^0)(i\gamma^0\gamma^3) = -\underbrace{\gamma^2\gamma^0\gamma^0\gamma^3} = -\underbrace{\gamma^2\gamma^3} = -(-1) = 1$$

$\{\gamma^0, \gamma^0\} = 2g^{00} = 2$
 $\{\gamma^2, \gamma^2\} = 2g^{22} = -2$

b)

$$\begin{aligned}
 \underline{C\gamma^\mu C^{-1}} &= i\gamma^2\gamma^0\gamma^\mu i\gamma^0\gamma^3 \\
 &= -\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^3 \\
 &= -\gamma^2\gamma^0\{2g^{\mu 0} - \gamma^0\gamma^\mu\}\gamma^3 \\
 &= -2g^{\mu 0}\gamma^2\gamma^0\gamma^3 + \gamma^2\gamma^0\gamma^0\gamma^\mu\gamma^3 \\
 &= 2g^{\mu 0}\gamma^2\gamma^2\gamma^0 + \gamma^2\gamma^\mu\gamma^3 \\
 &= -2g^{\mu 0}\gamma^0 + \gamma^2\{2g^{2\mu} - \gamma^2\gamma^\mu\} \\
 &= \underline{-2g^{\mu 0}\gamma^0 + 2g^{2\mu}\gamma^2 + \gamma^\mu} \quad (J)
 \end{aligned}$$

$$\left. \begin{aligned}
 \mu=0 &\rightarrow C\gamma^0 C^{-1} = -\gamma^0 = -(\gamma^0)^T \\
 \mu=1 &\rightarrow C\gamma^1 C^{-1} = +\gamma^1 = -(\gamma^1)^T \\
 \mu=2 &\rightarrow C\gamma^2 C^{-1} = -\gamma^2 = -(\gamma^2)^T \\
 \mu=3 &\rightarrow C\gamma^3 C^{-1} = +\gamma^3 = -(\gamma^3)^T
 \end{aligned} \right\} \begin{array}{l} \text{Som er} \\ \text{att).} \\ (K) \end{array}$$

c)

Adjuinkerte Dirac ligning:

$$\bar{\psi} [\gamma^\mu (-i\partial_\mu - eA_\mu) - m] = 0$$

Transponere:

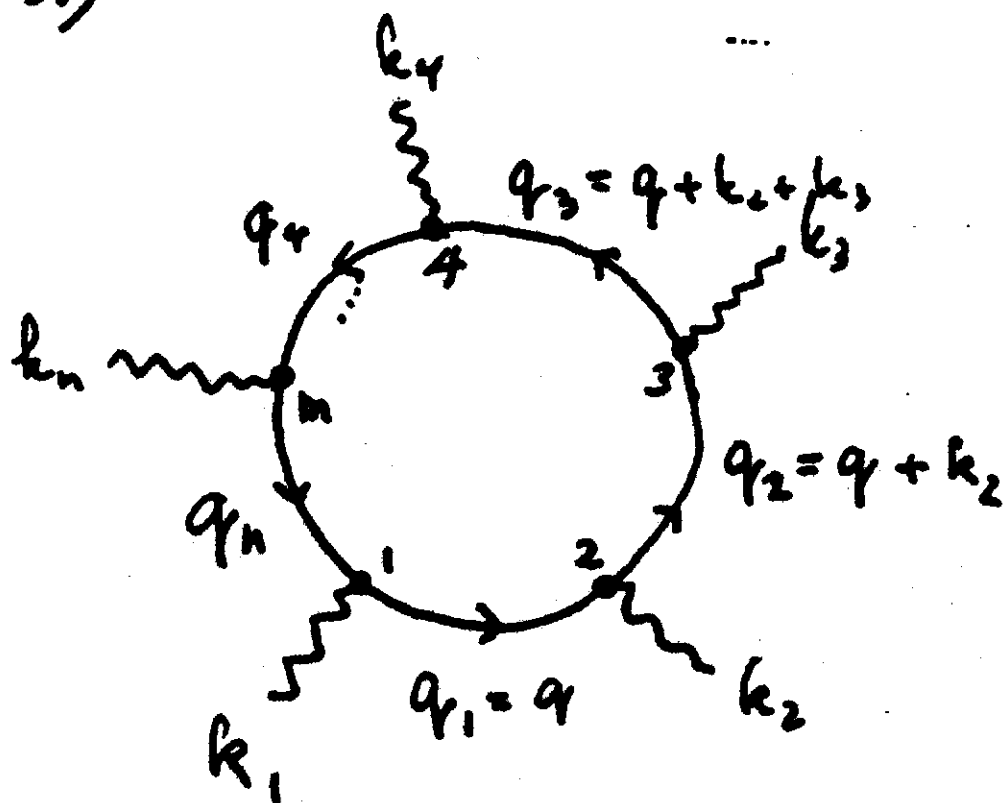
$$C | \quad [\gamma^{\mu T} (i\partial_\mu + eA_\mu) + m] \bar{\psi}^T = 0$$

$$[C \gamma^{\mu T} C^{-1} (i\partial_\mu + eA_\mu) + m] C \bar{\psi}^T = 0$$

$$\underbrace{\quad} = -\gamma^\mu \text{ fra ligning (17)}$$

$$\underline{[\gamma^\mu (i\partial_\mu + eA_\mu) - m] C \bar{\psi}^T = 0} \quad (2)$$

d)



Alle momenter k_i er positive i retningen inn til løkken.

Momentet som strømmer mellom punkt i og $i+1$:

$$q_i = q \quad \text{hvis } i=1$$

$$q_i = q + \sum_{j=2}^i k_j \quad \text{hvis } i \geq 2$$

Videre har vi følgende

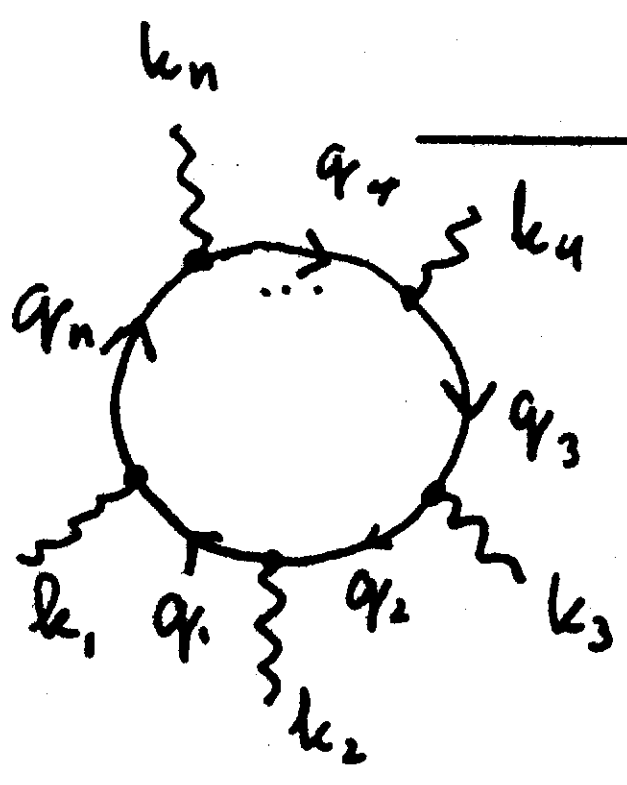
$$\sum_{i=1}^n k_i = 0$$

Ni: nu droppet foton polariseringen, siden
 disse ikke nødvendiggjør en ytre

Så er $\mathcal{M}_n = (-1) \int \frac{dq}{(2\pi)^4} \text{tr} (ie\gamma^\alpha iS_F(q_1) \dots$

$\dots ie\gamma^\alpha iS_F(q_n)$

(M)



definisjon av q_i som tidligere

fortegn på q_i som
 fri.

$\mathcal{M}_n = (-1) \int \frac{dq}{(2\pi)^4} \text{tr} (ie\gamma^\alpha iS_F(-q_n) \dots$

$ie\gamma^\alpha iS_F(-q_1))$

(N)

e)

Feynman propagatoren: $S_F(q) = \frac{\gamma^\mu q_\mu + m}{q^2 - m^2 + i\epsilon}$

$C S_F(q) C^{-1} = \frac{C \gamma^\mu C^{-1} q_\mu + m}{q^2 - m^2 + i\epsilon} = \frac{-\gamma^\mu q_\mu + m}{q^2 - m^2 + i\epsilon}$

$= S_F^T(-q)$ (O)

$\text{tr} (A^T B^T C^T \dots N^T) = A_{ij}^T B_{jk}^T C_{kl}^T \dots N_{mi}^T$

$= A_{ji} B_{kj} C_{lk} \dots N_{im} = N_{im} \dots C_{lk} B_{kj} A_{ji}$

$= \text{tr} (N \dots CBA)$ (P)

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$$\underline{\mathcal{M}_b} = (-1) \int \frac{d^4 q}{(2\pi)^4} \text{tr} (ie\gamma^{\alpha_1} iS_F(-q_n) \dots ie\gamma^{\alpha_n} iS_F(-q_1))$$

$$= (-1) \int \frac{d^4 q}{(2\pi)^4} \text{tr} (\overset{=1}{\underbrace{C^{-1}C}} ie\gamma^{\alpha_1} C^{-1}C iS_F(-q_n) C^{-1}C \dots$$

$$\dots \overset{=1}{\underbrace{C^{-1}C}} ie\gamma^{\alpha_n} \overset{=1}{\underbrace{C^{-1}C}} iS_F(-q_1))$$

$$= (-1) \int \frac{d^4 q}{(2\pi)^4} \text{tr} (ie(-\gamma^{\alpha_1})^T iS_F^T(q_n) \dots$$

$$\dots ie(-\gamma^{\alpha_n})^T iS_F^T(q_1))$$

$$= (-1)^n (-1) \int \frac{d^4 q}{(2\pi)^4} \text{tr} (ie\gamma^{\alpha_1 T} iS_F^T(q_n)$$

$$\dots ie(\gamma^{\alpha_n})^T iS_F^T(q_1))$$

$$= (-1)^n (-1) \int \frac{d^4 q}{(2\pi)^4} \text{tr} (iS_F(q_1) ie\gamma^{\alpha_n} \dots$$

$$\dots ie\gamma^{\alpha_1})$$

$$= (-1)^n (-1) \int \frac{d^4 q}{(2\pi)^4} \text{tr} (ie\gamma^{\alpha_1} iS_F(q_1) \dots$$

$$\dots ie\gamma^{\alpha_n} iS_F(q_n))$$

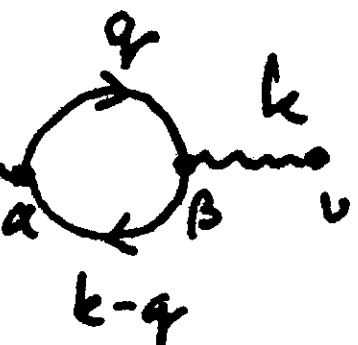
$$\underline{\underline{= (-1)^n \mathcal{M}_a}} \quad (Q)$$

done skulls via.

Oppgave 3

a)

$$\mu \text{---} \nu = i D_F^{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2 + i\epsilon}$$

$$\mu \text{---} \nu + \mu \text{---} \alpha \text{---} \beta \text{---} \nu = i D_F^{\mu\nu}(k)$$


$$i D_F^{\mu\nu}(k) = i D_F^{\mu\nu}(k) + i D_F^{\mu\alpha}(k) (-1) \int \frac{dq}{(2\pi)^4}$$

$$tr(i e \gamma_\alpha i S_F(q) i e \gamma_\beta i S_F(q-k)) i D_F^{\beta\nu}(k) \quad (R)$$

$$\text{Likha:} \quad i \Pi_{\alpha\beta}(k) = (-1) \int \frac{dq}{(2\pi)^4}$$

$$tr(i e \gamma_\alpha i S_F(q) i e \gamma_\beta i S_F(q-k)) \quad (S)$$

$$b) \quad \mu \text{---} \nu + \mu \text{---} \alpha_1 \text{---} \alpha_2 \text{---} \nu + \mu \text{---} \alpha_1 \text{---} \alpha_2 \text{---} \alpha_3 \text{---} \nu + \dots$$

$$= \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} + \frac{(-ig^{\mu\alpha_1})}{k^2 + i\epsilon} i a^2 g^{\alpha_1\alpha_2} \frac{(-ig^{\alpha_2\nu})}{k^2 + i\epsilon}$$

$$+ \left(\frac{-ig^{\mu\alpha_1}}{k^2 + i\epsilon} \right) i a^2 g^{\alpha_1 \alpha_2} \left(\frac{-ig^{\alpha_2 \alpha_3}}{k^2 + i\epsilon} \right) i a^2 g^{\alpha_3 \alpha_4} \left(\frac{-ig^{\alpha_4 \nu}}{k^2 + i\epsilon} \right) + \dots$$

$$= -ig^{\mu\nu} \left(\frac{1}{k^2 + i\epsilon} + \frac{1}{k^2 + i\epsilon} a^2 \frac{1}{k^2 + i\epsilon} + \frac{1}{k^2 + i\epsilon} a^2 \frac{1}{k^2 + i\epsilon} a^2 \frac{1}{k^2 + i\epsilon} + \dots \right)$$

$$= \frac{-ig^{\mu\nu}}{k^2 - a^2 + i\epsilon} \quad (T)$$

a represents some eu masse.

$$c) \quad i \Pi_{\alpha\beta}(k) = (-1) \int \frac{d^4 q}{(2\pi)^4} \text{tr} (i\epsilon\gamma_\alpha iS_F(q) i\epsilon\gamma_\beta iS_F(q-k))$$

$\sim q^3 dq$ $\sim \frac{1}{q}$ $\sim \frac{1}{(q-k)}$
 $\rightarrow \frac{1}{q}$
 near $k \rightarrow 0$

Altho

$$i \Pi_{\alpha\beta}(k \rightarrow 0) \sim \int \frac{q^3 dq}{q \cdot q} = \int q dq$$

som divergen.

a^2 is diverging!