NTNU Trondheim, Institutt for fysikk

Examination for FY3464 Quantum Field Theory II
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Allowed tools: mathematical tables
Some formulas can be found at the end of p.2.

1. Concepts.
Please answer *concise!*
a.) Give an explanation of the Goldstone theorem and how it can be avoided. (4 pts)
b.) Assume that a local conservation law of the type,

$$\partial_\mu T^{\mu\cdots\sigma} = 0\,, \tag{1}$$

for a tensor T of rank n holds in a classical field theory.
i) Explain why this relation can be broken on the quantum level. (4 pts)
ii) Give 2 examples for symmetries broken on the quantum level. (2 pts)
iii) (Extra question & extra points:) Give the necessary condition that the (classical) local conservation law (1) can be extended to a (classical) global conservation law in the presence of gravity. [(+2) pts]
c.) Sketch the beta-function $\beta(\mu)$ and the running coupling constant $\lambda(\mu)$, $g_s(\mu)$ of a scalar $\lambda\phi^4$ theory and QCD, respectively. (4 pts)
d.) Give one of the defining properties of an instanton. (2 pts)
e.) Explain, why or why not the contribution of the muon to loop corrections decouples in i) QED, and ii) in the standard model for $m_\mu \to \infty$. (4 pts)

a.) The spontaneous breaking of a (global) continuous symmetry leads to the appearance of massless Goldstone bosons. If the symmetry is gauged, these Goldstone bosons are converted into the longitudinal degrees of freedom of massive gauge bosons.

b.) i) The integration measure $\mathcal{D}\phi_i$ of a collection of field ϕ_i can change non-trivially under the classical symmetry; the equation $\partial_\mu T^{\mu\cdots\sigma} = 0$ translated into an operator equation is ambiguous; any possible regularization procedure breaks the symmetry.
ii) chiral anomaly, breaking of scale invariance.
iii) The presence of gravity implies that space-time becomes a (pseudo-) Riemannian manifold with a non-trivial metric tensor. In this case, we can integrate and obtain a global conservation law in general only, if $T^{\mu\cdots\sigma}$ is completely anti-symmetric (i.e. a differential form). A particular case is a conserved vector current. For a symmetric tensor as the energy-momentum tensor integration is only possible, if the manifold admits a Killing vector field.

c.) Figures...

d.) An instanton can be defined as i) (anti-) selfdual solution of the Euclidean Yang-Mills equation, $F = \pm\tilde{F}$, or ii) the non-trivial solution with finite action of the Euclidean Yang-Mills equation and smallest energy, or iii) a mapping $S^3 \to S^3$ with winding number $\nu = \pm 1$.

e.) The decoupling theorem of Appelquist-Corazone states that all effects of heavy particles X should disappear in loop corrections, if the theory after integrating out X stays renormalizable and unitary. i) QED is renormalizable and unitary for an arbitrary number of fermions, thus taking the limit $m_\mu \to \infty$ is ok, and the effect of a heavy muon vanishes.
ii) In the electroweak sector, $m_\mu \to \infty$ breaks SU(2) and perturbative unitarity (Yukawa coupling $y_\mu \to \infty$). As a result, heavy fermions do not decouple from certain loop corrections (e.g. to m_W).

2. Renormalisation of the $\lambda\phi^4$ theory at finite temperature.
Consider the usual $\lambda\phi^4$ theory for a scalar field ϕ in $D = 1 + 3$ dimensions,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \tag{2}$$

and in the unbroken phase, i.e. for $m^2 > 0$ and $\langle\phi\rangle = 0$, at finite temperature $T > 0$.
a.) Determine the counter-term δm^2 required to make the 1-loop self-energy Σ well-defined. (No need to perform any sum/integral explicitly.) (5 pts)
b.) Derive the $\mathcal{O}(\lambda)$ correction, i.e. the 2-loop vacuum diagram, to the partition function $\ln(Z)$ of a scalar gas. Show that the renormalization of the self-energy performed in a.) is sufficient to make the *thermal* correction well-defined. Calculate an explicit expression for the correction in the high-temperature limit $T \gg m$.
(12 pts)

a.) The self-energy is given by

$$\Sigma = \frac{\lambda}{2}\Delta(t = 0, \vec{x} = 0) = \frac{\lambda}{2}\sum_{\vec{k}} \frac{1 + 2n_{\vec{k}}}{2\omega_{\vec{k}}}\,. \tag{3}$$

The $T > 0$ term is finite, because the thermal distribution function $n_{\vec{k}}$ falls off exponentially for $|\vec{k}| \gg T$. We eliminate the divergent $T = 0$ term adding an additional interaction term of the form $\delta m^2\phi^2$ to the Lagrangian. The counter term is chosen such that m corresponds to the physical mass. Thus

$$\delta m^2 = -\frac{\lambda}{2}\sum_{\vec{k}} \frac{1}{2\omega_{\vec{k}}}\,. \tag{4}$$

b.) The 2-loop vacuum contribution is given by

$$\ln Z_1 = \frac{\lambda}{4!}\langle\phi^4\rangle = \frac{\lambda}{8}\langle\phi^2\rangle\langle\phi^2\rangle = \frac{\lambda}{8}\left[\sum_{\vec{k}} \frac{1 + 2n_{\vec{k}}}{2\omega_{\vec{k}}}\right]^2\,. \tag{5}$$

Splitting into $T=0$, $T>0$ and mixed parts, we have

$$= \frac{\lambda}{8}\Big[\underbrace{\Big(\sum_{\vec{k}} \frac{1}{2\omega_{\vec{k}}}\Big)^2}_{T=0} + \underbrace{\Big(\sum_{\vec{k}} \frac{n_{\vec{k}}}{\omega_{\vec{k}}}\Big)^2}_{\text{T>0}} + \underbrace{2\Big(\sum_{\vec{k}} \frac{1}{2\omega_{\vec{k}}}\Big)\Big(\sum_{\vec{k}} \frac{n_{\vec{k}}}{\omega_{\vec{k}}}\Big)}_{\text{mixed term}} \Big]. \tag{6}$$

The divergence of the $T=0$ term (not of interest here) has to be cured by a 2-loop counter term to the vacuum energy. In contrast, the mixed term in the 2-loop vacuum diagram is canceled adding the contribution from the counter-term δm^2 determined at 1-loop. This an example for the general rule that overlapping divergences are cured by renormalization at lower order.
Thus the temperature dependent $\mathcal{O}(\lambda)$ correction to the partition function is given by the second term. Evaluating it in the high-temperature limit $T \gg m$, we can set $\omega_{\vec{k}} = |\vec{k}|$ obtaining

$$\sum_{\vec{k}} \frac{n_{\vec{k}}}{\omega_{\vec{k}}} = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\omega} \frac{1}{\mathrm{e}^{\beta\omega}-1} = \frac{T^2}{2\pi^2} \underbrace{\int_0^\infty \mathrm{d}x \frac{x}{e^x-1}}_{\pi^2/6} = \frac{T^2}{12} \tag{7}$$

and thus the $\mathcal{O}(\lambda)$ correction is

$$\ln(Z_1) = \frac{\lambda}{8}\Big(\frac{T^2}{12}\Big)^2 = \frac{\lambda}{1152} T^4 \,. \tag{8}$$

3. Effective action.
We consider again a single scalar field. The effective action $\Gamma[\phi_c]$ is defined as the Legendre transform of the generating functional $W[J]$ for connected Green functions,

$$\Gamma[\phi_c] = W[J] - \int \mathrm{d}^4y J(y)\phi(y) \tag{9}$$

with $\phi_c(x) = \frac{\delta W[J]}{\delta J(x)}$. (4 pts)
a.) Show that the classical field $\phi_c(x)$ corresponds to the vacuum expectation value of the field ϕ in the presence of the source J.
b.) Show that the effective action of a free scalar field becomes the action of the classical field, $\Gamma[\phi_c] = S[\phi_c]$. (12 pts)
c.) Show that $\Gamma^{(2)}$ is equal to the inverse propagator or inverse 2-point function,

$$\Gamma^{(2)}(x_1,x_2) \equiv \left.\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(x_1)\delta\phi_c(x_2)}\right|_{\phi_c=0} = \mathrm{i}[\mathcal{G}^{(2)}(x_1,x_2)]^{-1} \,.$$

d.) Give a brief argument why spontaneous symmetry breaking as in the standard model does not spoil renormalisability. (4 pts) a.) We use simply the definition of the classical field,

$$\phi_c(x) = \frac{1}{\mathrm{i}Z}\frac{\delta Z[J(y)]}{\delta J(x)} = \frac{1}{Z}\int \mathcal{D}\phi\phi(x)\exp\mathrm{i}\int d^4y(\mathcal{L}+J\phi) = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} = \langle\phi(x)\rangle_J \tag{10}$$

b.) For a free scalar field

$$W_0[J] = -\frac{1}{2}\int d^4xd^4x'\; J(x)\Delta_F(x-x')J(x'), \tag{11}$$

and

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = -\int \mathrm{d}^4x'\;\Delta(x-x')J(x'). \tag{12}$$

If we apply the Klein-Gordon operator to the classical field

$$\begin{aligned}(\Box+m^2)\phi_c(x) &= -\int \mathrm{d}^4x\,(\Box+m^2)\Delta_F(x-x')J(x') \\ &= \int \mathrm{d}^4x\delta(x-x')J(x') = J(x)\end{aligned}$$

we get a solution to the classical field equation. Using

$$\begin{aligned}\Gamma_0 &= W_0[J] - \int \mathrm{d}^4x\; J(x)\phi_c(x) \\ &= \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\; J(x)\Delta_F(x-x')J(x'),\end{aligned}$$

inserting the above expression for $J(x)$ and integrating partially, the desired equation follows,

$$\begin{aligned}\Gamma_0[\phi_c] &= \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\big[(\Box+m^2)\phi_c(x)\big]\Delta_F(x-x')\big[(\Box'+m^2)\phi_c(x')\big] \\ &= \frac{1}{2}\int \mathrm{d}^4x\phi_c(x)(\Box+m^2)\phi_c(x) = S[\phi_c]\,.\end{aligned}$$

c.) We write

$$\delta(x_1-x_2) = \frac{\delta\phi(x_1)}{\delta\phi(x_2)} = \int \mathrm{d}^4x\,\frac{\delta\phi(x_1)}{\delta J(x)}\,\frac{\delta J(x)}{\delta\phi(x_2)}$$

using the chain rule. Next we insert $\phi(x) = \delta W/\delta J(x)$ and $J(x) = -\delta\Gamma/\delta\phi(x)$ to obtain

$$\delta(x_1-x_2) = -\int \mathrm{d}^4x\,\frac{\delta^2 W}{\delta J(x)\delta J(x_1)}\,\frac{\delta^2\Gamma}{\delta\phi(x_2)\delta\phi(x_2)}\,.$$

Setting $J=\phi=0$, it follows

$$\int \mathrm{d}^4x\,\mathrm{i}G^2(x,x_1)\,\Gamma^2(x,x_2) = -\delta(x_1-x_2)$$

or $\Gamma^2(x_1, x_2) = \mathrm{i}G^{2\,-1}(x, x_1)$.

d.) We can use the effective action to calculate and to renormalize all quantities before we shift the fields. In this case the counter terms derived for the unbroken theory are sufficient. As shifting the fields change nothing, new terms like ϕ^3 cannot require new counter terms.

Useful formulas

$$G(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} Z[J]\bigg|_{J=0} . \tag{13}$$

$$\mathcal{G}(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)} \mathrm{i}W[J]\bigg|_{J=0} . \tag{14}$$

$$Z[J] = Z[0] \exp(\mathrm{i}W[J]) \tag{15}$$

$$W_0[J] = -\frac{1}{2}\langle J \Delta_F J \rangle \tag{16}$$

$$(\Box + m^2)\Delta_F(x - x') = -\delta(x - x') \tag{17}$$

$$\Delta(t = 0, \vec{x} = 0) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\omega_{\vec{k}}} \left(\frac{1}{2} + \left\langle a_{\vec{k}}^\dagger a_{\vec{k}} \right\rangle \right) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1 + 2n_{\vec{k}}}{2\omega_{\vec{k}}} \tag{18}$$

$$n_{\vec{k}} = \frac{1}{\mathrm{e}^{\beta\omega_{\vec{k}}} \pm 1} \tag{19}$$

$$\int_0^\infty \mathrm{d}x \frac{x}{e^x - 1} = \frac{\pi^2}{6} \tag{20}$$