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Examination for FY3464 Quantum Field Theory I

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Allowed tools: mathematical tables

1. The $\lambda\phi^3$ theory.

Consider the theory of a massive real scalar field ϕ and a $\lambda\phi^3$ self-interaction in $d = 6$ dimensions.

a.) Write down the Lagrange density $\mathscr{L}$ and explain your choice of signs and pre-factors.
. (6 pts)

b.) Write down the corresponding generating functional for disconnected and connected Green functions. How does one obtain connected Green functions? (3 pts)

c.) Determine the dimension of the field ϕ and of the coupling λ. (3 pts)

d.) Draw the Feynman diagram(s) and write down the analytical expression for the self-energy iΣ (i.e. the loop correction for the free propgator) at order $\mathcal{O}(\lambda^2)$ in momentum space. (4 pts)

e.) Determine the symmetry factor of iΣ. (3 pts)

f.) Calculate the self-energy iΣ using dimensional regularisation, split the result into a divergent pole term and a finite remainder. (14 pts)

a.) The free Lagrangian is

$$\mathscr{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

the relative sign is fixed by the relativistic energy-momentum relation, the overall sign by the requirement that the Hamiltonian is bounded from below. As the self-interaction is odd, adding $+\frac{\lambda}{3!}\phi^3$ or $-\frac{\lambda}{3!}\phi^3$ is equivalent: both choices will lead to an unstable vacuum.
The prefactor 1/2 of the kinetic term corresponds to "canonically normalised field", leading to the correct size of vacuum fluctuations.
The prefactor of the $\lambda\phi^3$ term can be chosen arbitrary, if the Feynman rule is adjusted accordingly: For $-\mathrm{i}\lambda$, we should choose $\mathscr{L}_I = -\frac{\lambda}{3!}\phi^3$.

b.) The generating functional $Z[J]$ of disconnected Green functions is obtained from the path integral by i) adding a linear coupling to an external source J, ii) taking the limit $t, -t' \to \infty$ with $m^2 - \mathrm{i}\varepsilon$,

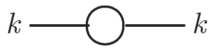
$$Z[J] = \langle 0|0\rangle_J = \mathcal{N}\int \mathcal{D}\phi \exp \mathrm{i}\int_\Omega \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3 + J\phi\right) = \exp \mathrm{i}W[J]\,.$$

The functional $W[J]$ generates connected Green functions,

$$G(x_1,\ldots,x_n) = \frac{1}{\mathrm{i}^n}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\mathrm{i}W[J]\bigg|_{J=0}\,. \tag{1}$$

c.) The action $S = \int \mathrm{d}^6 x \mathscr{L}$ has to be dimensionless. Thus $[\mathscr{L}] = m^6$, $[\phi] = m^2$, and thus the coupling is dimensionless, $[\lambda] = m^0$. [That's the reason why we do this exercise in $d = 6$.]

Using the Feynman rules gives for



in momentum space

$$\mathrm{i}\Sigma(k^2) = S\,(-\mathrm{i}\lambda)^2 \int \frac{\mathrm{d}^6 p}{(2\pi)^6} \frac{\mathrm{i}}{(p+k)^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

where the symmetry factor S is determined in e.) and the vertex $-\mathrm{i}\lambda$ was used.

e.) The self-energy is a second order diagram, corresponding to the term

$$\frac{1}{2!}\left(\frac{-\mathrm{i}\lambda}{3!}\right)^2 \int \mathrm{d}^4 y_1 \mathrm{d}^4 y_2 \langle 0|T[\phi(x_1)\phi(x_2)\phi^3(y_1)\phi^3(y_2) + (y_1 \leftrightarrow y_2)$$

in the perturbative expansion in coordinate space. The exchange graph $y_1 \leftrightarrow y_2$ is identical to the original one, canceling the factor $1/2!$ from the Taylor expansion. We count the number of possible ways to combine the fields in the time-ordered product into four propagators. We have three possibilities to contract $\phi(x_1)$ with a $\phi(y_1)$. Similiarly, there are three possibilities for $\phi(x_2)\phi(y_2)$. The remaining pairs of $\phi(y_1)$ and $\phi(y_2)$ can be contracted in 2! ways. Thus the symmetry factor is

$$S = \left(\frac{1}{2!} \times 2\right)\left(\frac{1}{3!}\right)^2 (3 \times 3 \times 2!) = \frac{1}{2}$$

[The symmetry factor is given for the vertex $-\mathrm{i}\lambda$.]

f.) We combine the two propagators (suppressing the $\mathrm{i}\varepsilon$) using (9),

$$\frac{1}{(p+k)^2 - m^2} \frac{1}{p^2 - m^2} = \int_0^1 \mathrm{d}x \frac{1}{D^2}$$

with

$$\begin{aligned} D &= x[(p+k)^2 - m^2] + (1-x)(p^2 - m^2) \\ &= (p+xk)^2 + x(1-x)k^2 - m^2 = q^2 + f\,, \end{aligned}$$

where we introduced $q = p + xk$ as new integration variable and set $f = x(1-x)k^2 - m^2$. We go now to $d = 2\omega = 6 - \varepsilon$ dimensions,

$$\mathrm{i}\Sigma(k^2) = \frac{1}{2}\lambda^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q+f)^2}.$$

Evaluating the integral with (10), using $\Gamma(2) = 1$ and $\omega = 3 - \varepsilon/2$ gives

$$\Sigma(k^2) = -\frac{\lambda^2}{2} \frac{\Gamma(-1+\varepsilon/2)}{(4\pi)^3} \int_0^1 \mathrm{d}x f \left(\frac{4\pi\mu^2}{f}\right)^{\varepsilon/2}.$$

Here, we added a mass scale μ in order to make the ε dependent term dimensionless such that we can expand it using (11),

$$\left(\frac{4\pi\mu^2}{f}\right)^{\varepsilon/2} = 1 + \frac{\varepsilon}{2}\ln\left(\frac{4\pi\mu^2}{f}\right) + \mathcal{O}(\varepsilon^2).$$

Expanding also

$$\Gamma(-1+\varepsilon/2) = -\left[\frac{2}{\varepsilon} + 1 - \gamma + \mathcal{O}(\varepsilon)\right]$$

we arrive at

$$\Sigma(k^2) = \frac{\alpha}{2}\left[\left(\frac{2}{\varepsilon} + 1 - \gamma\right)\left(\frac{k^2}{6} - m^2\right) + \int_0^1 \mathrm{d}x f \ln\left(\frac{4\pi\mu^2}{f}\right)\right]$$

where we used $\int_0^1 \mathrm{d}x f = k^2/6 - m^2$ and set $\alpha = \lambda^2/(4\pi)^3$. The obtained expression for the self-energy has the UV divergence isolated into an $1/\varepsilon$ pole which is ready for subtraction.

2. Fermions.

a.) Define left- and right-chiral fields ψ_L and ψ_R as eigenfunctions of γ^5. Express

$$\mathscr{L} = \bar\psi \mathrm{i}\partial\!\!\!/\psi - m\bar\psi\psi$$

in terms of ψ_L and ψ_R. (7 pts)

b.) Give an operator which commutes with the (free Dirac) Hamiltonian and can be used to classify the spin states of a fermion. Explain its meaning. (You don't have to calculate the commutator.) (3 pts)

a.) We can split any solution ψ of the Dirac equation into

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \equiv P_L\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1+\gamma^5)\psi \equiv P_R\psi. \tag{2}$$

Since $\gamma^5\psi_L = -\psi_L$ and $\gamma^5\psi_R = \psi_R$, $\psi_{L,R}$ are eigenfunctions of γ^5 with eigenvalue ±1. Expressing the mass term through these fields as

$$\bar\psi\psi = \bar\psi\left(P_L^2 + P_R^2\right)\psi = \psi^\dagger\left(P_R\gamma^0 P_L + P_L\gamma^0 P_R\right)\psi = \bar\psi_R\psi_L + \bar\psi_L\psi_R \tag{3}$$

and similarly for the kinetic term,

$$\bar\psi\partial\!\!\!/\psi = \bar\psi\left(P_L^2 + P_R^2\right)\partial\!\!\!/\psi = \psi^\dagger\left(P_R\gamma^0\gamma^\mu P_R + P_L\gamma^0\gamma^\mu P_L\right)\partial_\mu\psi = \bar\psi_L\partial\!\!\!/\psi_L + \bar\psi_R\partial\!\!\!/\psi_R, \tag{4}$$

the Dirac Lagrange density becomes

$$\mathscr{L} = \mathrm{i}\bar\psi_L\partial\!\!\!/\psi_L + \mathrm{i}\bar\psi_R\partial\!\!\!/\psi_R - m(\bar\psi_L\psi_R + \bar\psi_R\psi_L). \tag{5}$$

b.) One possibility is the helicity operator $h = \boldsymbol{s}\cdot\boldsymbol{p}/|\boldsymbol{p}|$, or more generally, $\gamma^5 s\!\!\!/$.

3. Scattering.
Derive the optical theorem

$$2\Im T_{ii} = \sum_n T_{in}^* T_{ni}.$$

Give a physical interpretation of this relation (less than 100 words). (7 pts)

The unitarity of the scattering operator, $S^\dagger S = SS^\dagger = 1$, expresses the fact that we (should) use a complete set of states for the initial and final states in a scattering process,

$$1 = \sum_n |n, +\infty\rangle \langle n, +\infty| = \sum_n S\,|n, -\infty\rangle \langle n, -\infty|\,S^\dagger = SS^\dagger. \tag{6}$$

We split the scattering operator S into a diagonal part and the transition operator T, $S = 1 + \mathrm{i}T$, and thus

$$1 = (1 + \mathrm{i}T)(1 - \mathrm{i}T^\dagger) = 1 + \mathrm{i}(T - T^\dagger) + TT^\dagger \tag{7}$$

or

$$\mathrm{i}TT^\dagger = T - T^\dagger. \tag{8}$$

We now consider matrix elements between the initial and final state,

$$\langle f|\,T - T^\dagger\,|i\rangle = T_{fi} - T_{if}^* = \mathrm{i}\,\langle f|\,TT^\dagger\,|i\rangle = \mathrm{i}\sum_n T_{fn}T_{in}^*. \tag{9}$$

If we set $|i\rangle = |f\rangle$, we obtain optical theorem as a connection between the forward scattering amplitude T_{ii} and the scattering into all possible states n,

$$2\Im T_{ii} = \sum_n |T_{in}|^2. \tag{10}$$

It relates the attenuation of a beam of particles in the state i, $\mathrm{d}N_i \propto -|\Im T_{ii}|^2 N_i$, to the probability that they scatter into all possible states n: what is lost, should show up somewhere.

4. Gauge invariance.
Consider a local gauge transformation

$$U(x) = \exp[\mathrm{i}g \sum_{a=1}^{m} \vartheta^a(x) T^a]$$

which changes a vector of fermion fields $\boldsymbol{\psi}$ with components $\{\psi_1, \ldots, \psi_k\}$ as

$$\psi(x) \to \psi'(x) = U(x)\psi(x)\,.$$

a.) Assume that U are elements of the non-abelian gauge group SU(n) and that $\{\psi_1, \ldots, \psi_5\}$ transform with the fundamental representation. What are then the values of n and m? What is the physical interpretation of m? (5 pts) b.) Derive the transformation law of $A_\mu = A_\mu^a T^a$ under a gauge transformation. One way to do this is to require that i) the covariant derivatives transform in the same way as ψ,

$$D_\mu \psi(x) \to [D_\mu \psi(x)]' = U(x)[D_\mu \psi(x)] .$$

and ii) that the gauge field should compensate the difference between the normal and the covariant derivative, (8 pts)

$$D_\mu \psi(x) = [\partial_\mu + \mathrm{i} g A_\mu(x)]\psi(x) .$$

c.) The non-abelian field-strength $F_{\mu\nu} = F_{\mu\nu}^a T^a$ transforms under a local gauge transformation $U(x)$ as (2 pts)

- ☐ $F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu}$
- ☐ $F_{\mu\nu} \to F'_{\mu\nu} = U(x) F_{\mu\nu} U^\dagger(x)$
- ☐ $F_{\mu\nu} \to F'_{\mu\nu} = U(x) F_{\mu\nu} U^\dagger(x) + \frac{\mathrm{i}}{g}(\partial_\mu U(x))\partial_\nu U^\dagger(x)$
- ☐ $F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} + [D_\mu, A_\nu]$

a.) The fundamental representation of SU(n) is n-dimensional. Since $\{\psi_1, \dots, \psi_5\}$ transforms with the fundamental representation, it is $n = 5$. Then $m = 5^2 - 1 = 24$ is the number of generators of SU(5), or more physically speaking, the number of gauge bosons.

b.) Combining both requirements gives

$$D_\mu \psi(x) \to [D_\mu \psi]' = U D_\mu \psi = U D_\mu U^{-1} U \psi = U D_\mu U^{-1} \psi', \tag{11}$$

and thus the covariant derivative transforms as $D'_\mu = U D_\mu U^{-1}$. Using its definition, we find

$$[D_\mu \psi]' = [\partial_\mu + \mathrm{i} g A'_\mu] U \psi = U D_\mu \psi = U[\partial_\mu + \mathrm{i} g A_\mu]\psi. \tag{12}$$

We compare now the second and the fourth term, after having performed the differentiation $\partial_\mu(U\psi)$. The result

$$[(\partial_\mu U) + \mathrm{i} g A'_\mu U]\psi = \mathrm{i} g U A_\mu \psi \tag{13}$$

should be valid for arbitrary ψ and hence after multiplying from the right with U^{-1} we arrive at

$$A_\mu \to A'_\mu = U A_\mu U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U) U^{-1} = U A_\mu U^{-1} - \frac{\mathrm{i}}{g} U \partial_\mu U^{-1}. \tag{14}$$

Here we also used $\partial_\mu(U U^{-1}) = 0$. For SU($n$), the gauge transformation U is an unitary transformation and one sets $U^{-1} = U^\dagger$.

c.) Option two

Some formulas

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \,. \tag{15}$$

$$\gamma^5 \equiv \gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \tag{16}$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{17}$$

$$\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 \tag{18}$$

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1-z)]^2} \,. \tag{19}$$

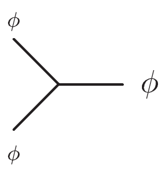
$$\begin{aligned} I(\omega,\alpha) &= \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 + 2pk + M^2 + \mathrm{i}\varepsilon]^\alpha} \\ &= \mathrm{i}\, \frac{(-\pi)^\omega}{(2\pi)^{2\omega}} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)} \frac{1}{[M^2 - p^2 + \mathrm{i}\varepsilon]^{\alpha-\omega}} \,. \end{aligned} \tag{20}$$

$$f^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2}\ln f + \mathcal{O}(\varepsilon^2) \,. \tag{21}$$

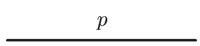
$$\Gamma(n+1) = n! \tag{22}$$

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \psi(n+1) + \mathcal{O}(\varepsilon)\right] \,, \tag{23}$$

$$\psi(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma \,, \tag{24}$$



$-\,\mathrm{i}\lambda$



$$\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$$