NTNU Trondheim, Institutt for fysikk

Examination for FY3464/8914 Quantum Field Theory I
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Allowed tools: mathematical tables

1. Miscellaneous and quiz

a.) Write down A^* for (3 pts)

$$A = \bar{u}(p_2)\gamma^\mu u(p_1)$$

b.) Calculate (3 pts)

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu].$$

c.) The covariant derivative of a Yang-Mills theory transforms under a local gauge transformation $U(x)$ as: (2 pt)

- ☐ $D_\mu \to D'_\mu = D_\mu$
- ☐ $D_\mu \to D'_\mu = U(x)D_\mu$
- ☐ $D_\mu \to D'_\mu = U(x)D_\mu U^\dagger(x)$
- ☐ $D_\mu \to D'_\mu = U(x)D_\mu U^\dagger(x) + \frac{\mathrm{i}}{g}(\partial_\mu U(x))U^\dagger(x)$

a.) Starting from

$$A^* = A^\dagger = (u^\dagger(p_2)\gamma^0\gamma^\mu u(p_1))^\dagger = u^\dagger(p_1)\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_2),$$

and using $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ and $(\gamma^0)^2 = 1$, we arrive at

$$A^* = \bar{u}(p_1)\gamma^\mu u(p_2).$$

b.) Contracting (1) with $\eta_{\mu\nu}$ gives

$$2\gamma^\mu\gamma_\mu = 2\eta^\mu_\mu = 8$$

or $\gamma^\mu\gamma_\mu = 4$. Together with $\mathrm{tr}(\mathbf{1}) = 4$ we find

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu] = 2\eta^{\mu\nu}\gamma_\mu\gamma_\nu - \gamma^\nu\gamma^\mu\gamma_\mu\gamma_\nu = -2\cdot 4\cdot 4 = -32\,.$$

c.) The covariant derivative of a Yang-Mills theory transforms homogenously under a local gauge, $D \to D'(x) = U(x)DU^\dagger(x)$.

d.) The field-strength of a Yang-Mills theory transforms homogenously under a local gauge, $\mathbf{F}(x) \to \mathbf{F}'(x) = U(x)\mathbf{F}(x)U^\dagger(x)$.

2. Scalar field.
Consider a real, scalar field ϕ with mass m and self-interaction $g\phi^3$.
a.) Write down the Lagrange density $\mathscr{L}$, explain your choice of signs and pre-factors (when physically relevant). (6 pts)
b.) Write down the generating functional for connected Green functions. (4 pts)
c.) Determine the mass dimension in $d = 4$ space-time dimensions of all quantities in the Lagrange density $\mathscr{L}$. (4 pts)
d.) Draw the divergent one-loop diagrams and determine their superficial degree of divergence D (in $d = 4$ space-time dimensions). (6 pts)
e.) Determine the number d of space-time dimensions for which the theory is renormalisable. (4 pts)

a.) The free Lagrangian is

$$\mathscr{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The relative sign is fixed by the relativistic energy-momentum relation, the overall sign by the requirement that the Hamiltonian is bounded from below. The factor 1/2 in the kinetic energy leads to "canonically normalised" field, the factor 1/2 for the mass follows then from the relativistic energy-momentum relation. As the self-interaction is odd, adding $+\frac{\lambda}{3!}\phi^3$ or $-\frac{\lambda}{3!}\phi^3$ is equivalent: both choices will lead to an unstable vacuum. In order to reproduce the Feynman rule, we used as normalisation the factor 1/3!,

$$\mathscr{L} = \mathscr{L}_0 - \frac{g}{3!}\phi^3.$$

b.) We set $m^2 \to m^2 - \mathrm{i}\varepsilon$ as damping term and add a source J coupled linearly to the field,

$$\mathscr{L}_{\text{eff}} = \mathscr{L} + \phi J\,. \tag{1}$$

The generating functional Z for disconnected Green functions is the path integral over fields of $\exp(\mathrm{i}\int d^4x\mathscr{L}_{\text{eff}}\})$,

$$Z[J] = \int \mathcal{D}\phi \, \exp\{\mathrm{i}\int d^4x\mathscr{L}_{\text{eff}}\} = \mathrm{e}^{\mathrm{i}W[J^\mu]}, \tag{2}$$

while $W[J^\mu]$ generates connected Green functions.

c.) The action $S = \int d^dx\mathscr{L}$ is (for $\hbar = 1$) dimensionless. The kinetic term $[(\partial\phi)^2] = m^4$ fixes the dimension of the field ϕ as m^1, consistent with the interpretation of m in the mass term as mass, $[m] = m^1$. This implies that the coupling g has the dimension $[g] = m^1$.

d.) The primitive divergent diagrams are the divergent 1-loop diagrams. We can order them by the number E of external (bosonic) legs and determine the superficial degree of divergence D by

naive power-counting, see the last page for the Feynman diagrams.
$E = 0$ and $D = 4$ corresponding a contribution to the cosmological constant,
$E = 1$ and $D = 2$ corresponding to a tadpole diagram,
$E = 2$ and $D = 0$ corresponding to the self-energy. Note that the vertex correction, $E = 3$, is already finite.
(The vacuum graphs ($E = 0$) are optional – you may prefer to "hide" them by asking for a properly normalized generating functional.)

e.) We have to find d such that $[g] = m^0$: For general d, it is $[\phi] = m^{(d-2)/2}$. Only solution for $[\phi^3] = d$ is thus $d = 6$ with $[\phi] = m^2$.

3. Fermion with Yukawa interaction.
Consider a Dirac fermion ψ with mass m interacting with real scalar field ϕ with mass M through a Yukawa interaction,

$$\mathscr{L} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi - \mathrm{i}g\bar{\psi}\gamma^5\psi\phi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2.$$

a.) Determine the global (internal) symmetries of the free, massless fermionic Lagrangian, $\mathscr{L} = \bar{\psi}\mathrm{i}\partial\!\!\!/\psi$, and the resulting Noether currents. (6 pts)
b.) Calculate the self-energy $\Sigma(p\!\!\!/)$ at one-loop of a fermion with momentum $p^2 \neq m^2$ using dimensional regularisation. Express $\Sigma(p\!\!\!/)$ as (12 pts)

$$\Sigma(p\!\!\!/) = \frac{A}{\varepsilon} + B\ln(D/\mu^2)]\,.$$

c.) What is your interpretation of the functional form of A? (3 pts)
d.) What is your interpretation of the dependence of the self-energy on the parameter μ? [c.) and d.): max. 50 words explanation.] (3 pts)

a.) Consider global phase transformations: First $U_V(1)$, **change** *phi* **to** ϑ

$$\psi(x) \to \psi'(x) = \mathrm{e}^{\mathrm{i}\phi}\psi(x) \qquad \text{and} \qquad \bar{\psi}(x) \to \bar{\psi}'(x) = \mathrm{e}^{-\mathrm{i}\phi}\bar{\psi}(x),$$

keep the Lagrangian invariant, $\delta\mathscr{L} = 0$. Noether's theorem (12) leads then with $\delta\psi = \mathrm{i}\psi$ to

$$j^\mu = \frac{\delta\mathscr{L}}{\delta(\partial_\mu\psi)}\,\delta\psi + \frac{\delta\mathscr{L}}{\delta(\partial_\mu\bar{\psi})}\,\delta\bar{\psi} = \bar{\psi}\mathrm{i}\gamma^\mu\mathrm{i}\psi + 0. \tag{3}$$

Thus the vector current is conserved. Next look at axial transformations $U_A(1)$,

$$\psi'(x) \to \mathrm{e}^{\mathrm{i}\phi\gamma^5}\psi(x) \quad \text{and} \quad \bar{\psi}(x) \to \bar{\psi}'(x) = (\mathrm{e}^{\mathrm{i}\phi\gamma^5}\psi(x))^\dagger\gamma^0 = \bar{\psi}(x)\mathrm{e}^{\mathrm{i}\phi\gamma^5}. \tag{4}$$

The resulting (infinitesimal) change is

$$\mathscr{L}' = \bar{\psi}'\mathrm{i}\partial\!\!\!/\psi' = \bar{\psi}(1 + \mathrm{i}\phi\gamma^5)\mathrm{i}\partial\!\!\!/(1 + \mathrm{i}\phi\gamma^5)\psi = \bar{\psi}(1 - \mathrm{i}\phi)\mathrm{i}\partial\!\!\!/(1 + \mathrm{i}\phi)\psi = \tag{5}$$

and thus again $\delta\mathscr{L} = 0$ (for $m = 0$). With $\delta\psi = \mathrm{i}\gamma^5\psi$ to

$$j^\mu = \frac{\delta\mathscr{L}}{\delta(\partial_\mu\psi)}\,\delta\psi = \bar{\psi}\mathrm{i}\gamma^\mu\mathrm{i}\gamma^5\psi \tag{6}$$

Thus the axial-vector current is conserved too (for $m = 0$).

b.) Following the fermion line and using the Feynman rules, we have

$$\mathrm{i}\Sigma(\not{p}) = (-\mathrm{i}g)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\gamma^5\frac{\mathrm{i}}{\not{p}+\not{k}-m}\gamma^5\frac{\mathrm{i}}{k^2-M^2}\,.$$

We combine first the denominators and complete then the square,

$$D = [(p+k)^2 - m^2]z + (k^2-M^2)(1-z) = k^2 + 2p\cdot kz + (p^2-m^2)z - M^2(1-z) = \tag{7}$$
$$= (k+zp)^2 + p^2z(1-z) - m^2z - M^2(1-z) \equiv q^2 + a \tag{8}$$

Next we evaluate the nominator using $\not{p}\gamma^5 = -\gamma^5\not{p}$ and $(\gamma^5)^2 = 1$, and substitute then k$\to q$,

$$N = \gamma^5(\not{p}+\not{k}-m)\gamma^5 = -(\not{p}+\not{k}+m) = -(\not{p}(1-z)+\not{q}+m)\,.$$

The linear term will vanish after integration and we drop it. Adding the mass scale μ^{4-n} to g and using ,,,, we find

$$\mathrm{i}\Sigma(\not{p}) = g^2(\mu^2)^{4-d}\mathrm{i}\,\frac{1}{(4\pi)^\omega}\,\frac{\Gamma(2-\omega)}{\Gamma(2)}\int_0^1\mathrm{d}z\frac{-(\not{p}(1-z)+m)}{a^{2-\omega}}$$

From the dimensionless quantity $(a/4\pi\mu^2)^{-\varepsilon}$, and expand $\Gamma(\varepsilon)$ and $(a/4\pi\mu^2)^{-\varepsilon}$ for small ε,

$$\Sigma(\not{p}) = -\frac{g^2}{16\pi^2}\left[(\not{p}-m/2)\frac{1}{\varepsilon} - \int_0^1\mathrm{d}z(\not{p}(1-z)+m)\ln(a/(4\pi\mu^2)\right]$$

c.) The coefficient of the divergent $1/\varepsilon$ term is a polynomial in the external momentum. More precisely, they correspond to terms $\bar{\psi}\mathrm{i}\partial\psi$ and $m\bar{\psi}\psi$ in the classical Lagrangian, and can thus be subtracted by mass and wave-function renormalisation.

d.) running parameters

4. Spin-1 fields.

a.) A massive spin-1 field A_μ satisfies the Proca equation,

$$(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)\,A_\nu + m^2A^\mu = 0\,.$$

Use the tensor method to determine the propagator $D_{\mu\nu}(k)$ of such a field [don't care about the poles]. (8 pts)
b) Give one argument why this method does not work setting $m = 0$. (3 pts)

a.) We write frist $m^2 A^\mu = m^2 \eta^{\mu\nu} A_\nu$. The propagator $D_{\mu\nu}$ for a massive spin-1 field is determined by

$$\left[\eta^{\mu\nu}(\Box + m^2) - \partial^\mu \partial^\nu\right] D_{\nu\lambda}(x) = \delta^\mu_\lambda \delta(x) . \tag{9}$$

Inserting the Fourier transformation of the propagator and the delta function gives

$$\left[\left(-k^2 + m^2\right) \eta^{\mu\nu} + k^\mu k^\nu\right] D_{\nu\lambda}(k) = \delta^\mu_\lambda . \tag{10}$$

We will apply the tensor method to solve this equation: In this approach, we use first all tensors available in the problem to construct the required tensor of rank 2. In the case at hand, we have at our disposal only the momentum k_μ of the particle—which we can combine to $k_\mu k_\nu$—and the metric tensor $\eta_{\mu\nu}$. Thus the tensor structure of $D_{\mu\nu}(k)$ has to be of the form

$$D_{\mu\nu}(k) = A\eta_{\mu\nu} + Bk_\mu k_\nu \tag{11}$$

with two unknown scalar functions $A(k^2)$ and $B(k^2)$. Inserting this ansatz and multiplying out, we obtain

$$\begin{aligned} \left[(-k^2 + m^2)\eta^{\mu\nu} + k^\mu k^\nu\right] \left[A\eta_{\nu\lambda} + Bk_\nu k_\lambda\right] &= \delta^\mu_\lambda , \\ -Ak^2\delta^\mu_\lambda + Am^2\delta^\mu_\lambda + Ak^\mu k_\lambda + Bm^2 k^\mu k_\lambda &= \delta^\mu_\lambda , \\ -A(k^2 - m^2)\delta^\mu_\lambda + (A + Bm^2)k^\mu k_\lambda &= \delta^\mu_\lambda . \end{aligned} \tag{12}$$

In the last step, we regrouped the LHS into the two tensor structures δ^μ_λ and $k^\mu k_\lambda$. A comparison of their coefficients gives then $A = -1/(k^2 - m^2)$ and

$$B = -\frac{A}{m^2} = \frac{1}{m^2(k^2 - m^2)} .$$

Thus the massive spin-1 propagator follows as

$$D^{\mu\nu}_F(k) = \frac{-\eta^{\mu\nu} + k^\mu k^\nu/m^2}{k^2 - m^2 + \mathrm{i}\varepsilon} . \tag{13}$$

b.) There's a mismatch of degrees of freedom, $3 \leftrightarrow 2$, between the massive and massless case/The longitudinal part $k^\mu k^\nu/m^2$ which blows up for $m \to 0$ does not contribute to the massless propagator/The projection operator following from the Maxwell Lagrangian has an eigenvalue 0 and is thus not invertible.

Feynman rules and useful formulas

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} . \tag{14}$$

$$\{\gamma^\mu, \gamma^5\} = 0 \quad \text{and} \quad (\gamma^5)^2 = 1 . \tag{15}$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{16}$$

$$\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0 \tag{17}$$

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}z}{[az + b(1-z)]^2}\,. \tag{18}$$

$$\int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2 - m^2 + \mathrm{i}\varepsilon]^\alpha} = \mathrm{i}\, \frac{(-1)^\alpha}{(4\pi)^\omega} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)} [m^2 - \mathrm{i}\varepsilon]^{\omega-\alpha}\,. \tag{19}$$

$$f^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2} \ln f + \mathcal{O}(\varepsilon^2)\,. \tag{20}$$

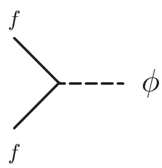
$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^{z-1} \tag{21}$$

$$\Gamma(n+1) = n! \tag{22}$$

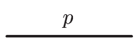
$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\varepsilon} + \psi_1(n+1) + \mathcal{O}(\varepsilon) \right], \tag{23}$$

$$\psi_1(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma\,, \tag{24}$$

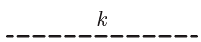
$$j^\mu = \frac{\delta \mathscr{L}}{\delta \partial_\mu \phi_a} \delta\phi_a - K^\mu. \tag{25}$$



$-\mathrm{i}g\gamma^5$



$$\frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\varepsilon}$$



$$\frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\varepsilon}$$