

TFY 4165 } Termisk fysikk
FY 1005 }

Kont. eksamen 07.08.2017

Løsningsforslag

①

1a) Stasjonære forhold $\Rightarrow \vec{\nabla} \cdot \vec{j} = 0$

$$\vec{j}(r) = -\kappa \vec{\nabla} T = j(r) \hat{e}_r.$$

$$j(r) = -\kappa \frac{dT}{dr}$$

$$\dot{Q} = \oint d\vec{A} \cdot \vec{j} = 2\pi r \cdot L \cdot j(r)$$

(Hv r L betyder av cylinderen)

$$\vec{\nabla} \cdot \vec{j} = 0 \Rightarrow \oint d\vec{A} \cdot \vec{j} = \dot{Q}(r) - \dot{Q}(r+dr) = 0$$

$$\frac{d\dot{Q}}{dr} = 0 \Rightarrow \dot{Q}(r) \text{ uavhengig av } r.$$

b) $\dot{Q} = 2\pi r L j(r) = -2\pi r L \kappa \frac{dT}{dr}$

$$dT = - \frac{\dot{Q}}{2\pi \kappa L} \frac{dr}{r}$$

$$T(r) - T(R_1) = - \frac{\dot{Q}}{2\pi L \kappa} \ln\left(\frac{r}{R_1}\right)$$

$\dot{Q}$ fastlegges ved $T(R_2) = T_2 \Rightarrow$

$$- \frac{\dot{Q}}{2\pi \kappa L} = \frac{T_2 - T_1}{\ln\left(\frac{R_2}{R_1}\right)} \Rightarrow$$

(2)

$$T(r) = T_1 + (T_2 - T_1) \frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)}$$

c) $T_2 = 300 \text{ K}$

Vi ser på effekt-løst pt. langede-ekt
lang cylinder.

$$T_1 = T_2 + \frac{(\dot{Q}/L)}{2\pi\kappa} \ln\left(\frac{R_2}{R_1}\right)$$

$$= 300 + \frac{1000}{2\pi \cdot 200} \ln(6)$$

$$= \underline{\underline{301.4 \text{ K}}}$$

(3)

2a) $C_y = \left(\frac{\partial H}{\partial T} \right)_y$

Oppgitt: $\left(\frac{\partial H}{\partial y} \right)_T = T \left(\frac{\partial X}{\partial T} \right)_y - X$

$$\left(\frac{\partial C_y}{\partial y} \right)_T = \frac{\partial^2 H}{\partial y \partial T} = \frac{\partial^2 H}{\partial T \partial y}$$

$$\begin{aligned} \frac{\partial^2 H}{\partial T \partial y} &= \left(\frac{\partial X}{\partial T} \right)_y + T \left(\frac{\partial^2 X}{\partial T^2} \right)_y - \left(\frac{\partial X}{\partial T} \right)_y \\ &= T \left(\frac{\partial^2 X}{\partial T^2} \right)_y = \left(\frac{\partial C_y}{\partial y} \right)_T \quad \text{Q.E.D.} \end{aligned}$$

Alternativt har vi bruke TDI direkte:

$$TdS = C_y dT + T \left(\frac{\partial X}{\partial T} \right)_y dy$$

$$dS = \left(\frac{\partial S}{\partial T} \right)_y dT + \left(\frac{\partial S}{\partial y} \right)_T dy$$

$$\left(\frac{\partial S}{\partial T} \right)_y = \frac{C_y}{T} \Rightarrow \frac{\partial^2 S}{\partial y \partial T} = \frac{1}{T} \left(\frac{\partial C_y}{\partial y} \right)_T$$

$$\left(\frac{\partial S}{\partial y} \right)_T = \left(\frac{\partial X}{\partial T} \right)_y \Rightarrow \frac{\partial^2 S}{\partial T \partial y} = \left(\frac{\partial^2 X}{\partial T^2} \right)_y$$

$$\Rightarrow \underline{\underline{\left(\frac{\partial C_y}{\partial y} \right)_T = T \left(\frac{\partial^2 X}{\partial T^2} \right)_y}}$$

(4)

b) $X = \chi \frac{y}{T}$

$$\left(\frac{\partial X}{\partial T}\right)_y = -\chi \frac{y}{T^2}$$

$$\left(\frac{\partial^2 X}{\partial T^2}\right)_y = 2\chi \frac{y}{T^3} \Rightarrow$$

$$\left(\frac{\partial C_y}{\partial y}\right)_T = 2\chi \frac{y}{T^2}$$

$$\underline{C_y = \chi \frac{y^2}{T^2}}$$

TDI langs adiabat ($TdS=0$)

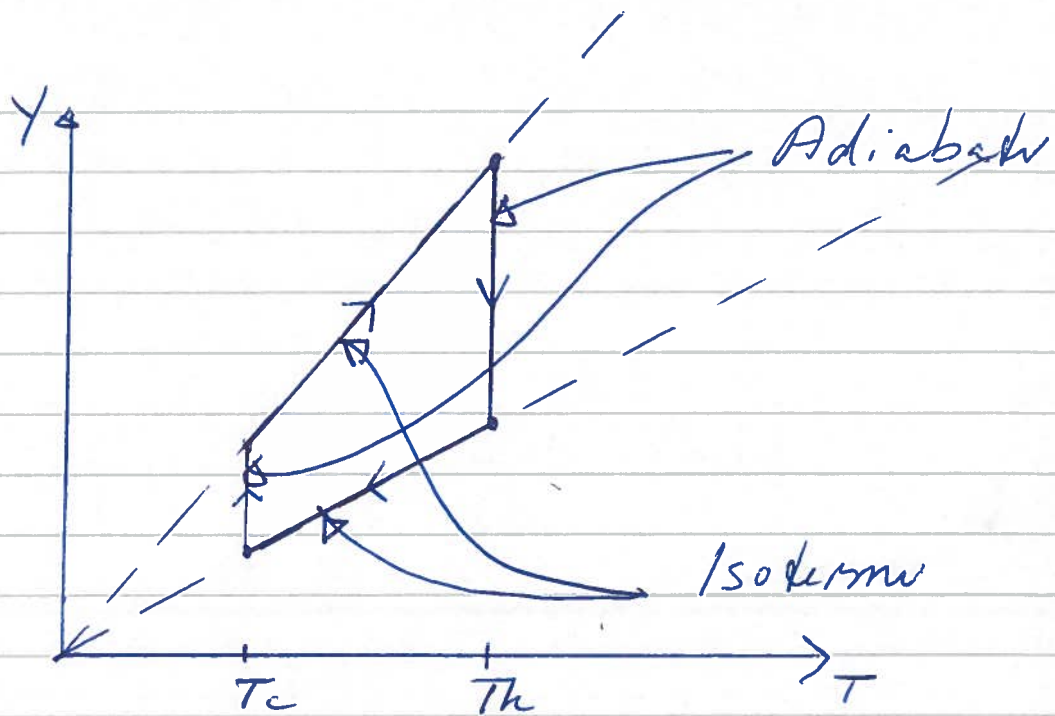
$$0 = \chi \frac{y^2}{T^2} dT - \chi \frac{y}{T} dy \quad | \cdot \frac{T}{\chi y^2}$$

$$0 = \frac{dT}{T} - \frac{dy}{y}$$

$$\ln\left(\frac{T}{y}\right) = \text{const.} \Rightarrow \underline{\underline{\frac{y}{T} = \text{const.}}} \quad \text{Q.E.D.}$$

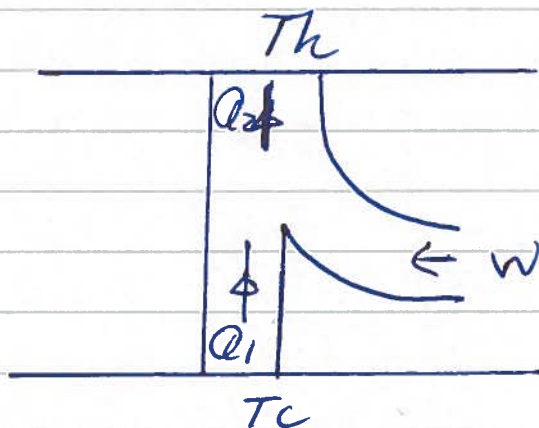
2c)

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2d)

Denne prosess kjører nå i
revers som en varmepumpe



$$\eta_V = \frac{|Q_2|}{W} = \frac{|Q_2|}{|Q_1 + Q_2|}$$

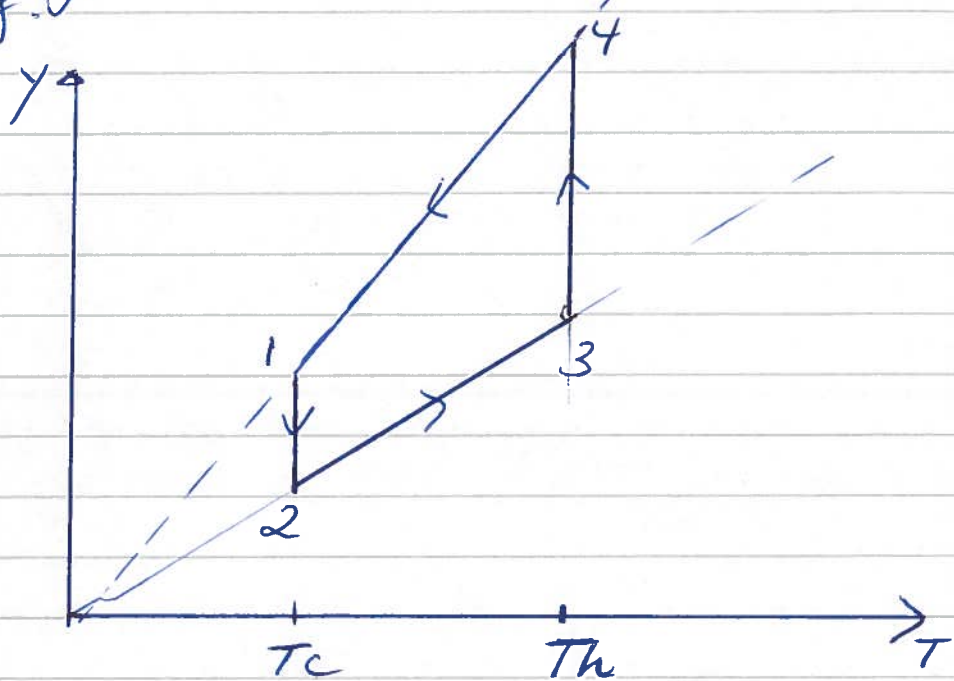
Q_1 = varme som tilføres system
ved T_c

Q_2 = varme som avgis av system
ved T_h .

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La oss også vise dette ved direkte utregning.

Proses
kjørt i
retur



Varme tilføres ved T_c : Q_t

Varme avgis ved T_h : Q_a

$$Q_t = \int_1^2 T ds = \int_1^2 T \left(\frac{\partial x}{\partial T} \right)_y dy$$

$$= - \frac{T_c}{T_c^2} \chi \int_1^2 y dy$$

$$Q_t = - \frac{\chi}{2T_c} (y_2^2 - y_1^2) > 0 \quad (y_2 < y_1)$$

$$Q_a = - \frac{\chi}{2T_h} (y_4^2 - y_3^2) < 0 \quad (y_4 > y_3)$$

(6a)

$$\eta_V = \frac{1}{1 + \frac{Q_L}{Q_H}}$$

$$\frac{Q_L}{Q_H} = \frac{T_H}{T_C} \left(\frac{y_2^2 - y_1^2}{y_4^2 - y_3^2} \right)$$

Na bruku vi at $\frac{y}{T}$ = konst.
langs adiabat. Fra figur:

$$\left. \begin{array}{l} \frac{y_3}{T_H} = \frac{y_2}{T_C} \\ \frac{y_4}{T_H} = \frac{y_1}{T_C} \end{array} \right\} \quad \begin{array}{l} \frac{y_3}{y_2} = \frac{T_H}{T_C} \\ \frac{y_3}{y_4} = \frac{y_2}{y_1} \end{array}$$

$$\frac{Q_L}{Q_H} = - \frac{T_H}{T_C} \left(\frac{y_2}{y_3} \right)^2 \frac{\left(1 - \frac{y_1^2}{y_2^2} \right)}{\left(1 - \frac{y_4^2}{y_3^2} \right)}$$

$$= - \frac{T_H}{T_C} \left(\frac{T_C}{T_H} \right)^2 = - \frac{T_C}{T_H}$$

$$\eta_V = \frac{1}{1 - \frac{T_C}{T_H}}$$

3a)

$$Z = \frac{1}{h^{2N} N!} Z_1^N$$

⑦

$$Z_1 \equiv Q_1 \cdot Z_0^2$$

$$Q_1 \equiv \int d^3r e^{-\frac{1}{2} m \omega^2 r^2}$$

$$Z_0 = \int_{-\infty}^{\infty} dp_x e^{-\frac{\beta p_x^2}{2m}}$$

$$= \sqrt{\frac{2m\pi}{\beta}}$$

$$Q_1 = 2\pi \int_0^R dr r e^{-\frac{\beta}{2} m \omega^2 r^2}$$

$$= \pi \int_0^{R^2} dv e^{-\frac{\beta m \omega^2}{2} v}$$

$$= -\frac{2\pi}{\beta m \omega^2} \left[e^{-\frac{\beta m \omega^2}{2} v} \right]_0^{R^2}$$

$$Q_1 = \frac{2\pi}{\beta m \omega^2} \left(1 - e^{-\frac{\beta m \omega^2 R^2}{2}} \right)$$

$$Z = \frac{1}{h^{2N} N!} (2\pi m k_B T)^N \cdot Q_1^N \quad (8)$$

Termisk de Broglie
bølglængde

$$= \frac{Q_1^N}{\lambda^{2N} N!} ; \quad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

b) $U = - \frac{\partial \ln Z}{\partial \beta}$

$$= +2N \frac{\partial \ln \lambda}{\partial \beta} - N \frac{\partial \ln Q_1}{\partial \beta}$$

$\beta m \omega^2 R^2 \gg 1 \Rightarrow$ vi ignorerer 2. led
i $Q_1 \Rightarrow$

$$Q_1 \approx \frac{2\pi}{\beta m \omega^2} \Rightarrow$$

$$U = 2N \frac{\partial \ln \lambda}{\partial \beta} - N \frac{\partial \ln \left(\frac{1}{\beta}\right)}{\partial \beta}$$

$$= \frac{N}{\beta} + \frac{N}{\beta} = \underline{\underline{2Nk_B T}}$$

$$\underline{\underline{C_V = 2Nk_B}}$$

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Klassisk ekvipartitionsprincip

$\frac{k_B}{2}$ bidrag til C_v for hver

uafhængig kvadratisk frihedsgrad.

Hv: 4 kvadratiske frihedsgrader
pr. partikel $\Rightarrow$

$$C_v = N \cdot 4 \cdot \frac{k_B}{2} = \underline{\underline{2Nk_B}}$$

c) $p = Nk_B T \frac{\partial \ln Q_1}{\partial V}$

$\beta m \omega^2 R^2 \gg 1 \Rightarrow Q_1$ uafhængig af $R \Rightarrow$

$$\underline{\underline{p = 0}}$$

d) $\beta m \omega^2 R^2 \ll 1 \Rightarrow$

$$Q_1 \approx \frac{2\pi}{\beta m \omega^2} \left(1 - \frac{\beta m \omega^2 R^2}{2} + \dots \right)$$

$$= \pi R^2 = V$$

$$p = Nk_B T \frac{\partial \ln V}{\partial V} = \underline{\underline{\frac{Nk_B T}{V}}}$$

Ma. gaur 19:

(10)

$$p = N k_B T \frac{\partial \ln Q_1}{\partial (\pi R^2)}$$

$$= \frac{N k_B T}{\pi} \frac{e^{-\frac{\beta m \omega^2 R^2}{2}}}{\left(1 - e^{-\frac{\beta m \omega^2 R^2}{2}}\right)} \frac{\beta m \omega^2}{2}$$

$$= \frac{m \omega^2}{2\pi} \frac{e^{-\frac{\beta m \omega^2 R^2}{2}}}{1 - e^{-\frac{\beta m \omega^2 R^2}{2}}} \cdot N$$

Izj n kar vi de gaur $\frac{\beta m \omega^2 R^2}{2} \ll 1 \Rightarrow$

$$p \approx \frac{m \omega^2}{2\pi} \frac{1}{\frac{\beta m \omega^2 R^2}{2}} N$$

$$= \frac{N k_B T}{\pi R^2} = \frac{N k_B T}{\checkmark}$$

For fullstendigheten skyld får vi
øye med det precise uttrykket i
motsetning gitt: $\frac{\beta m \omega^2 R^2}{2} \gg 1$ (11)

$$p \approx N \frac{m \omega^2}{2\pi} e^{-\frac{\beta m \omega^2 R^2}{2}}$$

Som viser at trykket partiklene utøver
mot vegg forsvinner eksponentielt med
volumet $V = \pi R^2$ og beholdes.

Grunn til at vi får det velkjente
ideell-gass resultatet i gassen

$$\frac{\beta m \omega^2 R^2}{2} \ll 1, \text{ og at i denne}$$

gassen vil partiklene være så
nære origo, like tiden at effekter
av det harmoniske potensialet
ikke merkes.