

11.12.2017Oppgave 1

a) $\vec{\nabla} \cdot \vec{j} = 0 \Rightarrow \oint d\vec{A} \cdot \vec{j} = 0$

$$P(r) = 4\pi r^2 j_r(r)$$

$$-P(r) + P(r+dr) = 0 \Rightarrow \frac{dP}{dr} = 0$$

 $P(r)$ uavh. av r

$$P = 4\pi r^2 j_r = -4\pi \kappa r^2 \frac{dT}{dr}$$

$$dT = - \frac{P}{4\pi \kappa} \frac{dr}{r^2}$$

Inneste kuleskall: $R_1 < r < R_2$

$$T(r) - T_1 = - \frac{P}{4\pi \kappa_1} \left(\frac{1}{R_1} - \frac{1}{r} \right)$$

Ytterste kuleskall: $R_2 < r < R_3$

$$T(r) - T_2 = - \frac{P}{4\pi \kappa_2} \left(\frac{1}{R_2} - \frac{1}{r} \right)$$

$t \rightarrow R_2^- :$

$$T_2 - T_1 = - \frac{P}{4\pi\kappa_1} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$P = Q_1 (T_1 - T_2) \Rightarrow$$

$$\underline{\underline{\frac{1}{Q_1} = \frac{1}{4\pi\kappa_1} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)}}$$

$t \rightarrow R_3^- :$

$$T_3 - T_2 = - \frac{P}{4\pi\kappa_2} \left(\frac{1}{R_2} - \frac{1}{R_3} \right)$$

$$P = Q_2 (T_2 - T_3) \Rightarrow$$

$$\underline{\underline{\frac{1}{Q_2} = \frac{1}{4\pi\kappa_2} \left(\frac{1}{R_2} - \frac{1}{R_3} \right)}}$$

$$b) \quad P = G_1 (T_1 - T_2) \\ = G_2 (T_2 - T_3)$$

$$\left. \begin{array}{l} T_1 - T_2 = \frac{P}{G_1} \\ T_2 - T_3 = \frac{P}{G_2} \end{array} \right\} \Rightarrow T_1 - T_3 = P \left(\frac{1}{G_1} + \frac{1}{G_2} \right) \\ = \frac{P}{G}$$

$$P = G (T_1 - T_3)$$

$$\underline{\frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2}}$$

Fourier's lov for det effektive
kuteskallet:

$$j_r = - \lambda_{\text{eff}} \frac{dT}{dr}; \quad R_1 < r < R_3$$

$$P = 4\pi r^2 j_r = - 4\pi \lambda_{\text{eff}} r^2 \frac{dT}{dr}$$

$$dT = - \frac{P}{4\pi \lambda_{\text{eff}}} \frac{dr}{r^2}$$

$$T(r) - T_1 = - \frac{P}{4\pi \lambda_{\text{eff}}} \left(\frac{1}{R_1} - \frac{1}{r} \right); \quad R_1 < r < R_3$$

$$r \rightarrow R_3^- :$$

$$T_3 - T_1 = - \frac{P}{4\pi\alpha_{eff}} \left(\frac{1}{R_1} - \frac{1}{R_3} \right)$$

$$P = Q (T_1 - T_3)$$

$$\frac{1}{Q} = \frac{1}{4\pi\alpha_{eff}} \left(\frac{1}{R_1} - \frac{1}{R_3} \right) = \frac{1}{Q_1} + \frac{1}{Q_2}$$

=>

$$\frac{1}{\alpha_{eff}} \left(\frac{1}{R_1} - \frac{1}{R_3} \right) = \frac{1}{\alpha_1} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{1}{\alpha_2} \left(\frac{1}{R_2} - \frac{1}{R_3} \right)$$

$$\frac{1}{\alpha_{eff}} \frac{1}{R_1} \left(1 - \frac{1}{4} \right) = \frac{1}{\alpha_1} \frac{1}{R_1} \left(1 - \frac{1}{2} \right) + \frac{1}{\alpha_2} \frac{1}{2R_1} \left(1 - \frac{1}{2} \right)$$

$$\frac{3}{4} \frac{1}{\alpha_{eff}} = \frac{1}{2\alpha_1} + \frac{1}{4\alpha_2}$$

$$\frac{3}{\alpha_{eff}} = \frac{2\alpha_2 + \alpha_1}{\alpha_1 \alpha_2}$$

$$\underline{\underline{\alpha_{eff} = \frac{3\alpha_1\alpha_2}{\alpha_1 + 2\alpha_2}}}$$

c) Ta faks ligning for $T(r)$
for innerste kuleskall; $r \rightarrow R_1$

$$T_2 - T_1 = - \frac{P}{G_1}$$

$$T_2 = T_1 - \frac{P}{G_1}$$

Fra ligning for ypperste kuleskall:

$$T_3 - T_2 = - \frac{P}{G_2}$$

$$T_2 = T_3 + \frac{P}{G_2}$$

Men vi har også: $P = G(T_1 - T_3)$

$$T_2 = T_1 - \frac{G}{G_1}(T_1 - T_3)$$

$$T_2 = T_3 + \frac{G}{G_2}(T_1 - T_3)$$

Her er nå alle størrelser kjente

Vi setter inn i G_1, G_2, G de

oppgitte verdier

$$R_2 = 2R_1; \quad R_3 = 4R_1$$

$$\alpha_1 = 2\alpha_2$$

$$G_1 = 8\pi \mathcal{A}_1 R_1$$

$$G_2 = 16\pi \mathcal{A}_2 R_1$$

$$\frac{G}{G_1} = \frac{G_2}{G_1 + G_2} = \frac{2\mathcal{A}_2}{2\mathcal{A}_2 + \mathcal{A}_1} = \frac{1}{2}$$

$$\frac{G}{G_2} = \frac{G_1}{G_1 + G_2} = \frac{\mathcal{A}_1}{\mathcal{A}_1 + 2\mathcal{A}_2} = \frac{1}{2}$$

Fra ligning for T_2 :

$$T_2 = T_1 - \frac{1}{2} (T_1 - T_3)$$

$$= 1000\text{ K} - 400\text{ K} = \underline{600\text{ K}}$$

Fra ligning for T_3 :

$$T_2 = T_3 + \frac{1}{2} (T_1 - T_3)$$

$$= 200\text{ K} + 400\text{ K} = \underline{600\text{ K}}$$

Oppgave 2

$$\begin{aligned} \underline{a)} \quad Z &= \prod_{j=1}^N \sum_{i=1}^2 e^{-\beta \epsilon_{ji}} \\ &= \prod_{j=1}^N (e^{\beta \epsilon} + e^{-\beta \epsilon}) \\ &= (e^{\beta \epsilon} + e^{-\beta \epsilon})^N = (2 \cosh \beta \epsilon)^N \end{aligned}$$

$$U = - \frac{\partial \ln Z}{\partial \beta}$$

$$= -N \frac{\partial \ln(2 \cosh \beta \epsilon)}{\partial \beta}$$

$$= -N \epsilon \frac{\sinh(\beta \epsilon)}{\cosh(\beta \epsilon)} = \underline{\underline{-N \epsilon \tanh(\beta \epsilon)}}$$

$$b) \quad \beta \epsilon \ll 1 \Rightarrow \tanh(\beta \epsilon) \approx \beta \epsilon$$

$$U = - \frac{N \epsilon^2}{k_B T}$$

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V = \underline{\underline{\frac{N \epsilon^2}{k_B T^2}}}$$

$$c) \quad T dS = C_V dT$$

Reversibel T-utjæwning, varmeisoleret system.

$$\Delta S = \Delta S_1 + \Delta S_2 = 0$$

1. lov:

$$0 = \Delta U + W_{\max} \Rightarrow W_{\max} = -\Delta U$$

$$\Delta U = \Delta U_1 + \Delta U_2$$

$$\Delta S_1 = \int_{T_1}^{T_S} \frac{C_V}{T} dT = \frac{N\varepsilon^2}{2k_B} \left(\frac{1}{T_1^2} - \frac{1}{T_S^2} \right) < 0$$

$$\Delta S_2 = \int_{T_2}^{T_S} \frac{C_V}{T} dT = \frac{N\varepsilon^2}{2k_B} \left(\frac{1}{T_2^2} - \frac{1}{T_S^2} \right) > 0$$

$$\Delta U_1 = \int_{T_1}^{T_S} C_V dT = \frac{N\varepsilon^2}{k_B} \left(\frac{1}{T_1} - \frac{1}{T_S} \right) < 0$$

$$\Delta U_2 = \int_{T_2}^{T_S} C_V dT = \frac{N\varepsilon^2}{k_B} \left(\frac{1}{T_2} - \frac{1}{T_S} \right) > 0$$

T_S bestemmes fra $\Delta S = 0$

$$\frac{N\varepsilon^2}{2k_B} \left(\frac{1}{T_1^2} + \frac{1}{T_2^2} - \frac{2}{T_S^2} \right) = 0$$

$$T_s^2 = \frac{2 T_1^2 T_2^2}{(T_1^2 + T_2^2)}$$

$$T_s = \frac{\sqrt{2} T_1 T_2}{\sqrt{T_1^2 + T_2^2}}$$

$$W_{\max} = -(\Delta U_1 + \Delta U_2)$$

$$= - \frac{N \epsilon^2}{k_B} \left(\frac{1}{T_1} + \frac{1}{T_2} - \frac{2}{T_s} \right)$$

$$= - \frac{N \epsilon^2}{k_B T_1 T_2} \left[T_1 + T_2 - \sqrt{2} \sqrt{T_1^2 + T_2^2} \right]$$

$$= \frac{N \epsilon^2}{k_B T_1 T_2} \left\{ \sqrt{2} \sqrt{T_1^2 + T_2^2} - (T_1 + T_2) \right\}$$

For fullstendighetens skyld vis vi også at dette arbeidet er positivt

$$f(x) = \frac{2(T_1^2 + T_2^2)}{(T_1 + T_2)^2} = \frac{2(1+x^2)}{(1+x)^2} ; 0 \leq x \leq 1$$

$$x = \frac{T_2}{T_1}$$

$$f(x) \geq 1 ; 0 \leq x \leq 1$$

$$f(x) = 2 ; x = 0$$

$$f(x) = 1 ; x = 1$$

$$\underline{W_{max} = \frac{N \epsilon^2}{k_B} \left(\frac{T_1 + T_2}{T_1 T_2} \right) [\sqrt{f(x)} - 1]}$$

$$f(x) \geq 1 \quad ; \quad 0 < x < 1$$

$$\Rightarrow$$

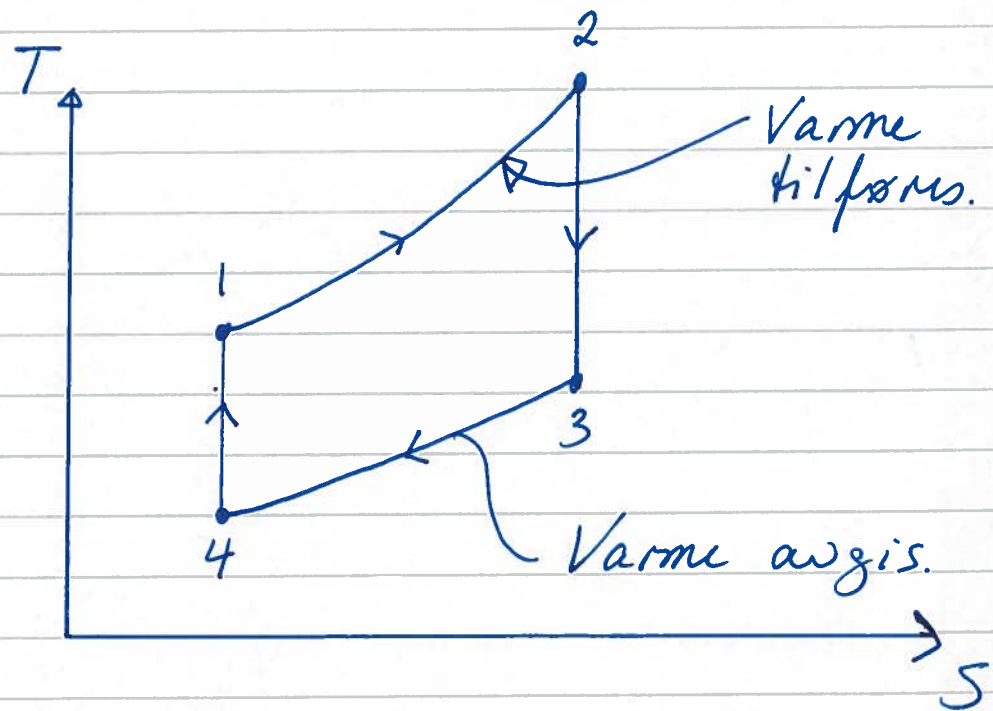
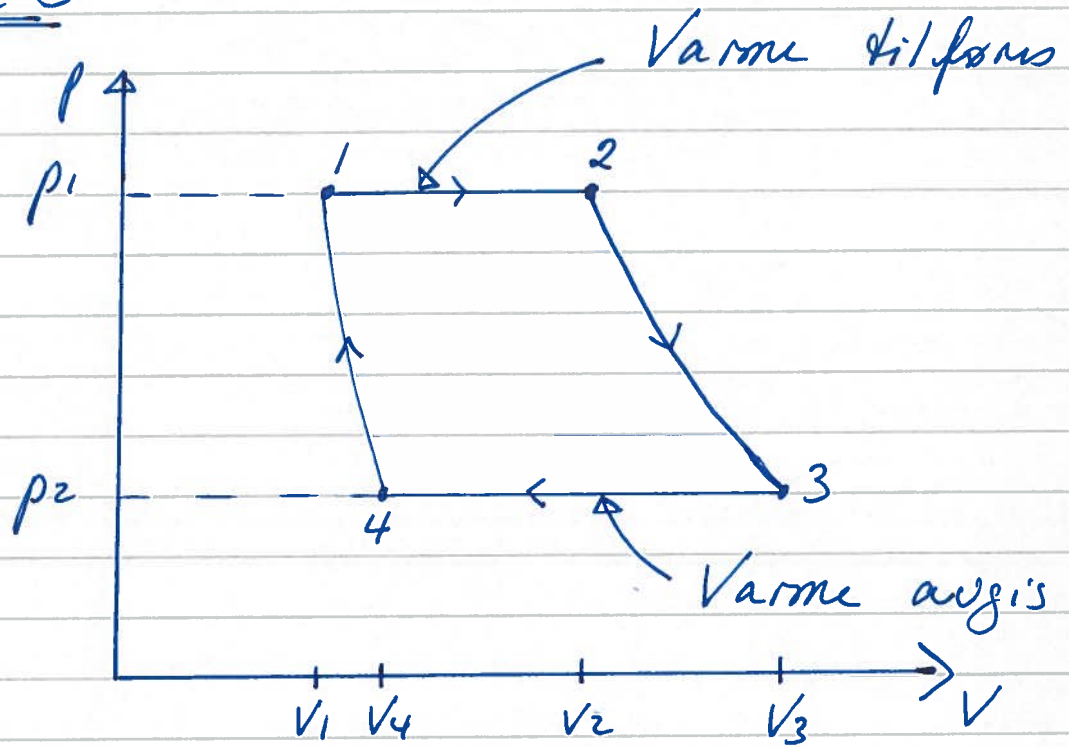
$$W_{max} \geq 0$$

$$W_{max} = 0 \quad \text{när} \quad f(x) = 1.$$

Det sker när $x=1$ d: $T_1 = T_2$

Oppgave 3

a)



b) Tilført varme : $1 \rightarrow 2$

Avgitt varme : $3 \rightarrow 4$

TDI: $TdS = C_p dT$

$$Q_t = C_p (T_2 - T_1) > 0$$

$$\underline{Q_a = C_p (T_4 - T_3) < 0}$$

$$W = Q_t + Q_a$$

$$= C_p (T_2 - T_1 + T_4 - T_3)$$

$$\eta = \frac{W}{Q_t} = 1 - \frac{|Q_a|}{Q_t}$$

$$= 1 - \left(\frac{T_3 - T_4}{T_2 - T_1} \right)$$

Adiabat ligning: Vi ønsker at
elimine T -er til fordel for p -er
og bruger derfor

$$T p^{\frac{1-\gamma}{\gamma}} = \text{const}$$

langs adiabatene $2 \rightarrow 3$ og $4 \rightarrow 1$.

$$2 \rightarrow 3: T_2 p_1^{\frac{1-\gamma}{\gamma}} = T_3 p_2^{\frac{1-\gamma}{\gamma}} \quad (\times)$$

$$4 \rightarrow 1: T_4 p_2^{\frac{1-\gamma}{\gamma}} = T_1 p_1^{\frac{1-\gamma}{\gamma}} \quad (\times \times)$$

$$\eta = 1 - \frac{T_4}{T_1} \frac{\left(\frac{T_3}{T_4} - 1\right)}{\left(\frac{T_2}{T_1} - 1\right)}$$

Fra (x) finner vi:

$$\frac{T_4}{T_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1-\gamma}{\gamma}} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Ved å dividere (x) med (xx) finner vi

$$\frac{T_3}{T_4} = \frac{T_2}{T_1} \Rightarrow \frac{\frac{T_3}{T_4} - 1}{\frac{T_2}{T_1} - 1} = 1$$

$$\eta = 1 - \frac{T_4}{T_1} = \underline{\underline{1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}}}$$

$\frac{\gamma-1}{\gamma}$ øker monotont med γ
 $\frac{p_2}{p_1} < 1 \Rightarrow \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$ minsker monotont

med $\gamma \Rightarrow \eta$ øker monotont med γ .

$$C_p^{(1)} = \frac{5}{2} N k_B$$

$$C_p^{(2)} = \frac{7}{2} N k_B$$

$$\gamma^{(1)} = 5/3$$

$$; \gamma^{(2)} = 7/5$$

$$\gamma^{(1)} > \gamma^{(2)} \Rightarrow$$

η er større for $\bar{n}$ -atomer
idell gas enn ϕ -atomer