

Exam, May 14, 2004: Suggested solution

Problem 1

a) System matrix:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & f_1 + f_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Transfer matrix:

$$\begin{pmatrix} 1 & d' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{f_2}{f_1} & f_1 + f_2 - \frac{f_2}{f_1}d - \frac{f_1}{f_2}d' \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

For conjugate planes we have $B' = 0$, and the transfer matrix reduces to the imaging matrix:

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ -P & 1/\beta \end{pmatrix} = \begin{pmatrix} -\frac{f_2}{f_1} & 0 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix},$$

where β is the magnification and P is the power.

b) From $B' = 0$ we obtain the image relation: $d' = \frac{f_2}{f_1}(f_1 + f_2) - \left(\frac{f_2}{f_1}\right)^2 d$.

The magnification is: $\beta = A' = -f_2/f_1 = -1/6$.

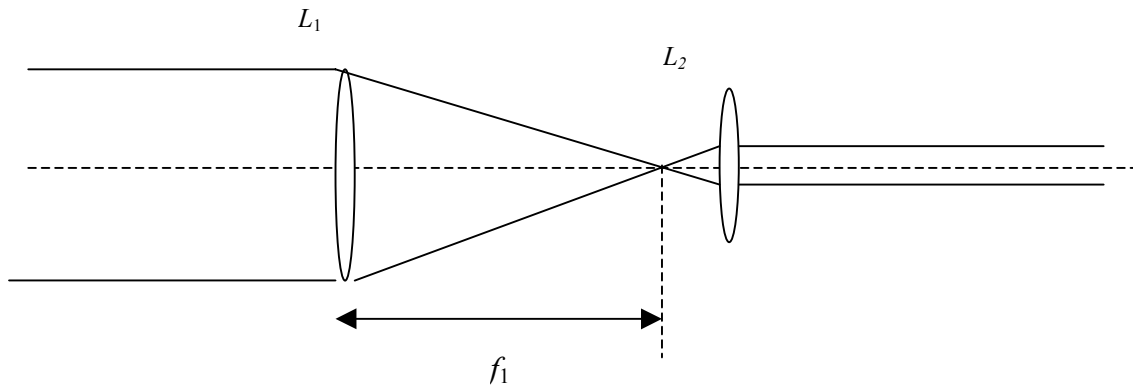
The principal planes H and H' are conjugate planes (images of each other) for which the magnification is: $\beta = +1$. Here the magnification is $\beta = -f_2/f_1 = -1/6$ for *all* conjugate planes. Therefore this system does not have principal planes.

We here have $P = 0$, so this is a so-called afocal system (a telescope).

c)

- The aperture stop is the physical stop that limits the ray bundle from an object point on axis to the corresponding image point.
- The entrance pupil is the aperture stop imaged to the object space (i.e., seen from the object side).
- The exit pupil is the aperture stop imaged to the image space (i.e., seen from the image side).

- A chief ray is the ray from an object point to the corresponding image point that passes through the center of the aperture stop (and the associated pupils).
- The field stop is the stop that limits the bundle of chief rays from the object to the image.
- The entrance window is the field stop imaged to the object space (i.e., seen from the object side).
- The exit window is the field stop imaged to the image space (i.e., seen from the image side).



For the ray bundle from an object point on-axis we now have the situation illustrated in the figure above, and we immediately see that L1 is the aperture stop and the entrance pupil. Since we only have two stops in the system, L2 is the field stop and the exit window.

The exit pupil is L1 imaged into image space by L2. It is located at the image distance

$$d_{EP}' \text{ where } \frac{1}{f_1 + f_2} + \frac{1}{d_{EP}'} = \frac{1}{f_2}, \text{ which yields: } d_{EP}' = \frac{f_2}{f_1} (f_1 + f_2) = \underline{5.83 \text{ cm to the}}$$

$$\underline{\text{right of } L_2} \text{ and has the diameter } D_{EP}' = D_1 \frac{d_{EP}'}{f_1 + f_2} = D_1 \frac{f_2}{f_1} = \underline{1.333 \text{ cm}}.$$

The entrance window is L2 imaged into object space by L1. It is located at the object

$$\text{distance } d_{EW} \text{ where } \frac{1}{f_1 + f_2} + \frac{1}{d_{EW}} = \frac{1}{f_1}, \text{ which yields: } d_{EW} = \frac{f_1}{f_2} (f_1 + f_2) = \underline{2.1 \text{ m to}}$$

$$\underline{\text{the left of } L_1} \text{ and its diameter is } D_{EW} = D_2 \frac{d_{EW}}{f_1 + f_2} = D_2 \frac{f_1}{f_2} = \underline{12 \text{ cm}}.$$

- d) This is a telescope. The observer looks at a virtual image with image height

$$y' = \beta y = -\frac{f_2}{f_1} y = -y/6. \text{ This image is both demagnified and inverted but it is}$$

observed from a distance that is much less than the original object distance. From the image relation in b) we see that the distance from the exit pupil to the image is:

$$d' - d_{EP}' = -\left(\frac{f_2}{f_1}\right)^2 d = -\beta^2 d.$$

The angle subtended by the object at the entrance pupil is y/d , but the corresponding angle subtended by the image at the exit pupil is $\left| \frac{y'}{d'-d_{EP}} \right| = -\frac{1}{\beta} \frac{y}{d}$. So although the magnification is $\beta = -1/6$, the visual magnification is: $1/\beta = -f_1/f_2 = -6$.

Problem 2

a) The coherence function is:

$$\begin{aligned} \Gamma(\tau) &= I_0 \int_{\omega_0 - \Delta\omega/2}^{\omega_0 + \Delta\omega/2} \exp(-i\omega\tau) d\omega = I_0 \frac{\exp[-i(\omega_0 + \Delta\omega/2)\tau] - \exp[-i(\omega_0 - \Delta\omega/2)\tau]}{-i\tau} \\ &= I_0 \exp(-i\omega_0\tau) \frac{\exp(-i\Delta\omega\tau/2) - \exp(i\Delta\omega\tau/2)}{-i\tau} \\ &= I_0 \Delta\omega \exp(-i\omega_0\tau) \frac{\sin(\Delta\omega\tau/2)}{\Delta\omega\tau/2} = I_0 \Delta\omega \exp(-i\omega_0\tau) \text{sinc}(\Delta\omega\tau/2). \end{aligned}$$

Substitution into the given formula yields:

$$I(s) = 2I_0 \Delta\omega \left[1 + \text{sinc}\left(\frac{\Delta\omega s}{2c}\right) \cos(\omega_0 s/c) \right]$$

and we see that the interference term is the product $\text{sinc}\left(\frac{\Delta\omega s}{2c}\right) \cos(\omega_0 s/c)$ where the sinc-term describes the envelope function. QED!

b) The visibility is defined as $V \equiv \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$, where I_{max} and I_{min} are, respectively, the maximum and minimum intensity as we move from one fringe to the next by changing the path-length difference.

In the interference signal $I(s) = 2I_0 \Delta\omega \left[1 + \text{sinc}\left(\frac{\Delta\omega s}{2c}\right) \cos(\omega_0 s/c) \right]$, the cosine term describes the interference fringes while the sinc-term is the slowly varying envelope that determines the visibility. We then have

$$I_{max} = 2I_0 \Delta\omega \left[1 + \left| \text{sinc}\left(\frac{\Delta\omega s}{2c}\right) \right| \right] \text{ and } I_{min} = 2I_0 \Delta\omega \left[1 - \left| \text{sinc}\left(\frac{\Delta\omega s}{2c}\right) \right| \right], \text{ from which we}$$

obtain the visibility function:

$$V(s) = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \left| \text{sinc}\left(\frac{\Delta\omega s}{2c}\right) \right|.$$

The first zero of the visibility function is at $\frac{\Delta\omega s}{2c} = \pi$, which yields for the longitudinal coherence length: $l_c = \underline{2\pi c / \Delta\omega}$.

- c) When we go from one fringe to the next, s is changed by the amount: $2\pi c / \omega_0 = \lambda_0$. The number of interference fringes that can be observed with good visibility in the range $-l_c / 2 \leq s \leq l_c / 2$ is therefore: $l_c / \lambda_0 = \underline{\omega_0 / \Delta\omega}$.

- d) From the given data we obtain: $l_c = 20\lambda_0 = \underline{10 \mu\text{m}}$, $\omega_0 = 2\pi c / \lambda_0 = \underline{3.77 \cdot 10^{15} \text{ s}^{-1}}$, and $\Delta\omega = \omega_0 / 20 = 6\pi \cdot 10^{13} \text{ s}^{-1} = \underline{18.85 \cdot 10^{13} \text{ s}^{-1}}$.

Problem 3

- a) From the given formulas the Fourier series expansion is given by:

$$t(y) = \sum_{m=-\infty}^{\infty} \frac{1}{a} T_0 \left(m \frac{2\pi}{a} \right) \exp \left(i m \frac{2\pi}{a} y \right),$$

$$T_0(v) = \mathcal{F}\{t_0(y)\} = \int_{-a/2}^{a/2} t(y) \exp(-ivy) dy.$$

We here have

$$T_0(v) = \int_{-d/2}^{d/2} \exp(-ivy) dy = d \frac{\sin(vd/2)}{vd/2} = d \text{sinc}(vd/2)$$

and direct substitution yields the given formula. QED!

- b) The field immediately behind the object can now be written as:

$$U(x, y, 0) = A_0 t(y) = \sum_{m=-\infty}^{\infty} A_0 \frac{d}{a} \text{sinc}(\pi m d / a) \exp(2\pi i m y / a).$$

A plane wave with amplitude A is given by:

$$A \exp(i\mathbf{k} \cdot \mathbf{r}) = A \exp[i(ux + vy + z\sqrt{k^2 - u^2 - v^2})]$$

For $z = 0$ it reduces to: $A \exp[i(u x + v y)]$. Each term in the sum above has precisely this

form: $A_m \exp[i(u_m x + v_m y)]$ with $A_m = A_0 \frac{d}{a} \text{sinc}(\pi m d / a)$, $u_m = 0$, and $v_m = 2\pi m / a$.

For $z \geq 0$ the m -th term in the Fourier series therefore gives rise to the plane wave:

$$A_m \exp(i\mathbf{k}_m \cdot \mathbf{r}) = A_0 \frac{d}{a} \text{sinc}(\pi m d / a) \exp[i(2\pi m y / a + z\sqrt{k^2 - (2\pi m / a)^2})]$$

- c) The m -th plane wave propagates at the angle θ_m with the axis. From the given formula we then have

$$A_m \exp[i(2\pi m y / a + z\sqrt{k^2 - (2\pi m / a)^2})] = A_m \exp[ik(y \sin \theta_m + z \cos \theta_m)]$$

and immediately see that $2\pi m / a = k \sin \theta_m$, which yields the well-known grating equation: $a \sin \theta_m = m\lambda$.

- d) Only a finite number of plane waves will be freely propagated because, for $|m| > a / \lambda$, we have so-called, evanescent plane waves that are exponentially damped in the z direction. With the given parameters we can have freely propagated waves for $|m| \leq a / \lambda = 20$, so that 41 plane waves may be freely propagated (one for $m=0$, 20 for $m<0$, and 20 for $m>0$). However, some of these plane waves may be absent (“missing orders”) because the amplitude A_m vanishes. With the given parameters we have:

$$A_m = A_0 \frac{1}{2} \text{sinc}(\pi m/2), \text{ which vanishes for all } m = \pm 2, \pm 4, \pm 6, \dots \text{etc.}$$

In this case we only have 21 freely propagated plane waves.