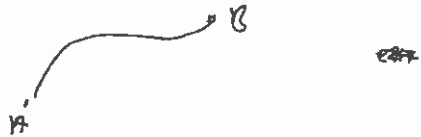


Section A

1. (a) Fermat principle - principle of least time.

A ray follows the path from a point A to a point B that takes the least time.

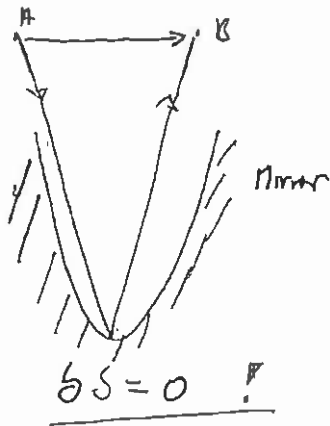
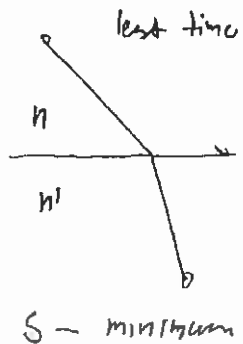


The more accurate/mathematical form of Fermat principle

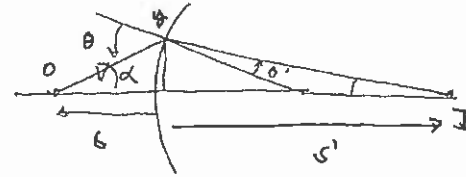
is that $S(A, B) = \int_A^B n(s) ds$ - is optimal,

i.e. that $\delta S = \delta \int_A^B n(s) ds = 0$

Examples



1b)



In deriving eqn. (A.1), several approximations are performed:

$$n \sin \theta = n' \sin \theta' \rightarrow n \theta \approx n' \theta',$$

i.e. the paraxial form of Snell's law.

Furthermore

$$\alpha \approx \frac{y}{s}$$

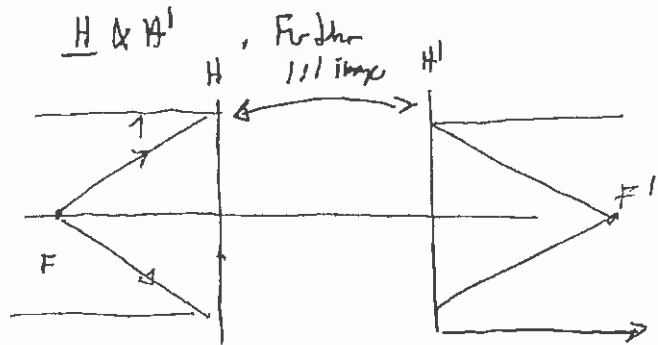
$$\alpha' \approx \frac{y}{s'}$$

} assuming just small angles
only valid for rays close
to the optic axis
(on-axis)

Other assumptions:

- spherical optics give perfect images
- no reflection losses, no scattering losses, no absorption losses
- no diffraction or size of lens/aperture.

1c) Principal planes defined by a unit magnification where



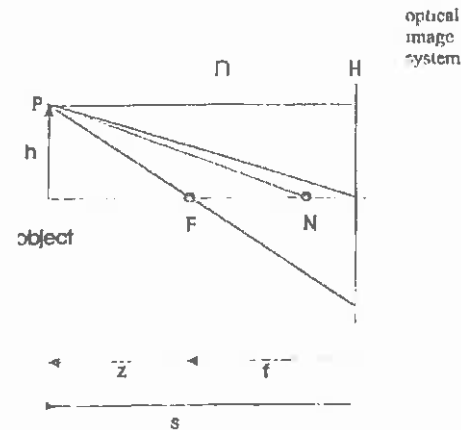
① rays from focal point F exits ∞ to other side

② rays from focal point F' enters ∞ to other side

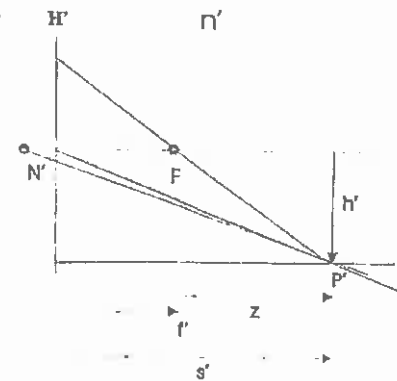
Gaussian lens eqn. $\frac{n}{s} + \frac{n'}{s'} = \frac{n'}{f}$ is valid

w.r.t. principal planes H & H'

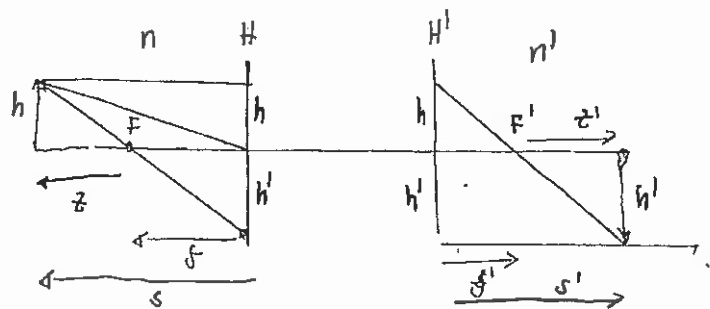
2)



optical
image
system



1d)



From the figure it is evident that similar triangles on object side & image side may be used to derive the relationship between s', s & f' .

$$\frac{h}{z} = -\frac{h'}{f} \quad , \quad \frac{h}{f'} = -\frac{h'}{z'} \quad (1)$$

hence

$$\frac{h'}{h} = -\frac{f}{z} = -\frac{z'}{f'} \quad \equiv \quad \text{transverse magnification } M_t \quad (2)$$

Newborn image equation is given by,

$$ff' = zz' \quad , \quad \text{in accordance with the appendix.} \quad (3)$$

To obtain the Gaussian image eqn: $\frac{n}{s} + \frac{n'}{s'} = \frac{n'}{f'}$, (4)
see Appendix, we set

$$z = (s-f) \quad \& \quad z' = (s'-f') \quad (5)$$

and insert into eqn. (3)

$$\Rightarrow \frac{ff'}{zz'} = (s-f)(s'-f') = ss' - fs' - sf' + ff' = zz' \quad (6)$$

$$\Rightarrow ss' = fs' + sf'$$

$$1 = \frac{f}{s} + \frac{f'}{s'} \quad (6)$$

$$\text{now we use that } \frac{n}{f} = \frac{n'}{f'} \quad , \quad \Rightarrow f = \frac{n f'}{n'} \quad (7)$$

the latter equation (7) comes from the definition of the principal planes and that repeated application of ray reflects surfaces always gives: $M_t \cdot M_a = \frac{n}{n'}$

with $M_t = 1$ between principal planes, we have that

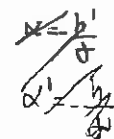
$$M_a = \frac{n}{n'} = \frac{\alpha'}{\alpha}$$

$$\alpha n = n' \alpha'$$

$$\text{hence } \frac{n}{f} = \frac{n'}{f'}$$

$$\Rightarrow 1 = \frac{n f'}{n' s} + \frac{f'}{s'}$$

$$\Rightarrow \frac{n'}{s'} + \frac{n}{s} = \frac{n'}{f'}$$



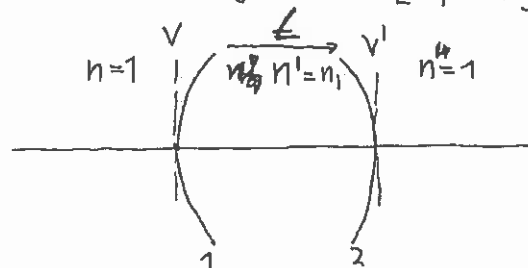
$$-\frac{FF'}{f} = \alpha$$

$$-\frac{H'F'}{f'} = \alpha' = -\frac{FF'}{f'}$$

$$M_t = \frac{h'}{h} = \frac{s'}{s} = \frac{(s'-f')}{(s-f)} = \frac{n'}{n} \frac{(s'-f')}{(s-f)}$$

Q2.a/ Prove that for thin lens:

$$p = \frac{1}{f} = (n-1) \left[\frac{1}{R_1} - \frac{1}{R_2} \right]$$



produce system transfer matrix between V & V'

note 2

$$M_{\text{transfer}} = \begin{bmatrix} 1 & 0 \\ \frac{n-n_1}{R_1} & \frac{n_1}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ +C_1 & \frac{n_1}{n} \end{bmatrix}$$

$$C_1 = \left(\frac{n-n_1}{R_1} \right) \quad \text{and} \quad C_2 = \left(\frac{n-n_1}{R_2} \right) = \left(\frac{n_1-1}{R_2} \right)$$

transfer matrix

$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

note 1

$$\begin{bmatrix} 1 & 0 \\ C_1 & \frac{n}{n_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C_1 & \frac{1}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & n_1 \end{bmatrix}$$

$$C_1 = \left(\frac{1-n_1}{R_1} \right)$$

$$M_{V'V} = \begin{bmatrix} 1 & 0 \\ C_1 & \frac{1}{n_1} \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C_2 & n_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ C_1 & \frac{1}{n_1} \end{bmatrix} \begin{bmatrix} 1+C_2L & Ln_1 \\ C_2 & n_1 \end{bmatrix}$$

$$\begin{bmatrix} 1+C_2L & Ln_1 \\ (1+C_2L)C_1 + \frac{C_2}{n_1} & Ln_1C_1 + 1 \end{bmatrix}$$

now for the equivalent, it is known that the refractive power is given by

$$p = \frac{1}{f'} = -C \quad , \text{ see table 18-2,}$$

hence

$$p = - \left((1+C_2L)C_1 + \frac{C_2}{n_1} \right) \quad , \quad L \rightarrow 0 \text{ for thin lens}$$

$$\Rightarrow -p = C_1 + \frac{C_2}{n_1} = \left(\frac{1-n_1}{R_1} \right) + \left(\frac{n_1-1}{R_2 n_1} \right)$$

$$= \left(\frac{n_1-1}{n_1} \right) \left(-\frac{1}{R_1} + \frac{1}{R_2} \right) \quad , \quad p = \left(\frac{n_1-1}{n_1} \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \quad \text{Q.E.D.}$$

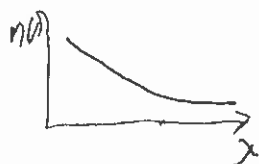
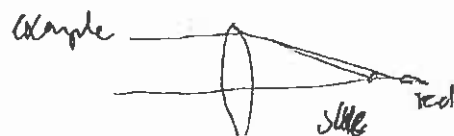
b) It is stated that

$$P = \frac{1}{f} = (n(\lambda) - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

Dispersion means that $n(\lambda)$ has a wavelength dependence.

Therefore $P(\lambda) = \frac{1}{f(\lambda)}$

Typically, $n(\lambda)$ of glass varies from $n \sim 1.5$ to 1.51 over a few hundred nm's, and causes light of different wavelengths (colors) to have a different focal point. This is ~~called~~^{denoted} chromatic aberration.



2c) This question is similar to question (1), set up

$$M_{H1} = M_{L2} M_{\text{lens}} M_{L1}$$

$$= \begin{bmatrix} 1 & 0 \\ -P_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P_1 & 1 \end{bmatrix}$$

lenses in air

$$\begin{bmatrix} 1 - LP_1 & L \\ -P_1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 - LP_1 & L \\ -P_2(1 - LP_1) - P_1 & -P_2L + 1 \end{bmatrix}$$

Using again the results from part 1b-2.

then $\frac{1}{f'} = P_1 + P_2 - LP_1P_2$, where f' denotes

or $\frac{1}{f'} = \frac{1}{f_1} + \frac{1}{f_2} - L \frac{1}{f_1} \frac{1}{f_2}$

the object plane on image side

d) $p = p_1 + p_2 - L p_1 p_2$

In order for the system to be aplanatic,
then we simply require that: (see e.g. Pedrotti p. 76.)
(see I's lecture)

$$\frac{\partial p}{\partial \lambda} = \frac{\partial p}{\partial n} \left(\frac{\partial n}{\partial \lambda} \right) = 0$$

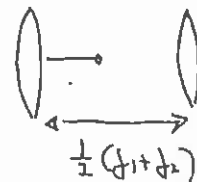
$$\frac{\partial p_1}{\partial \lambda} = \left(\frac{1}{k_{11}} - \frac{1}{k_{12}} \right) \cdot \frac{\partial n}{\partial \lambda} = \frac{1}{(n-1)} p_1 \left(\frac{\partial n}{\partial \lambda} \right)$$

$$\frac{\partial p_2}{\partial \lambda} = \frac{1}{(n-1)} p_2 \frac{\partial n}{\partial \lambda}$$

$$\begin{aligned} \Rightarrow \frac{\partial p}{\partial n} &= \frac{\partial p_1}{\partial n} + \frac{\partial p_2}{\partial n} - L \left(\left(\frac{\partial p_1}{\partial n} \right) p_2 + \frac{\partial p_2}{\partial n} p_1 \right) \\ &= \frac{p_1}{(n-1)} + \frac{p_2}{(n-1)} - \frac{L}{(n-1)} \left(\frac{p_1 p_2}{(n-1)} + p_2 p_1 \right) = 0 \end{aligned}$$

$$\begin{aligned} L &= \frac{p_1 + p_2}{2 p_1 p_2} = \frac{1}{2} \left[\frac{1}{p_2} + \frac{1}{p_1} \right] \\ &= \underline{\underline{\frac{1}{2} (f_2 + f_1)}} \end{aligned}$$

(cont'd) The system is thus aplanatic if the
two lenses are separated by $\frac{1}{2} (d_1 + d_2)$



$$\begin{aligned} \text{We note that } p^2 &= \frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{2} (f_1 + f_2) \frac{1}{f_1 f_2} \\ &= \frac{1}{2} \left(\frac{1}{f_1} + \frac{1}{f_2} \right) = \frac{1}{2} (p_1 + p_2) \end{aligned}$$

If both lenses are positive, then we have
a normal imaging system, but we recognise the
configuration found from the

- Huygens eye piece

and the

- Ramsden eye piece.

The latter are also in telescopes & microscopes,
expressions.

See Pedrotti page 77.

e) Achromatic lens: see It's learning-pocket note,
Pedrotti p. 458)

$$P = P_1 + P_2 - \underbrace{L}_{\text{lens}} P_1 P_2, \quad L \rightarrow 0 \text{ for compound lens}$$

$$\frac{\partial P}{\partial \lambda} = \frac{P_1}{(n_1-1)} \frac{dn_1}{d\lambda} + \frac{P_2}{(n_2-1)} \frac{dn_2}{d\lambda} = 0$$

$$\Rightarrow \frac{P_1}{(n_1-1)} \frac{dn_1}{d\lambda} = - \frac{P_2}{(n_2-1)} \frac{dn_2}{d\lambda}$$

$(n_1-1)(n_2-1) > 0$ always for transparent glass in the visible

$\Rightarrow P_2$ must have opposite power of P_1

$$\text{i.e. } \text{sign}(f_1) = - \text{sign}(f_2)$$

For $P = (P_1 + P_2) > 0$, then

$$P_1 > |P_2| \quad \text{if} \quad P_1 > 0$$

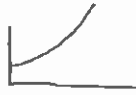
$$\text{or } P_1 = (P - P_2) \Rightarrow \frac{(P - P_2)}{(n_1-1)} \frac{dn_1}{d\lambda} = - \frac{P_2}{(n_2-1)} \frac{dn_2}{d\lambda}$$

$$\text{Let } K_1 = \frac{1}{(n_1-1)} \frac{dn_1}{d\lambda} \quad \text{and} \quad K_2 = \frac{1}{(n_2-1)} \frac{dn_2}{d\lambda}$$

$$\Rightarrow (P - P_2) K_1 = - P_2 K_2$$

$$\left. \begin{array}{l} P_1 + P_2 \\ P + P_2 \end{array} \right\} P_2 = \frac{P}{(K_1 - K_2)} < 0$$

$$\text{hence } K_2 > K_1$$

For a ^{dense} dispersive material 
 $\frac{dn}{d\lambda}$ is large

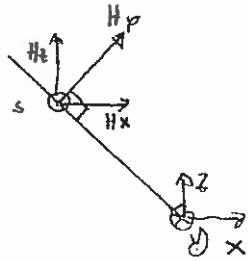
Hence high dispersion, $-P_2$, $-ve$ focal length
(typically crown)



low dispersion (typically flint)

$+P_1$, $+ve$ focal length

11a) $E(\vec{r}, t) = E_s(\vec{r}, t) \hat{e}_s = E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$



Use Faraday's law (Appendix, Maxwell's eqns.):
 $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad i(\vec{k} \cdot \vec{r} - \omega t)$
 Assume time harmonic solutions $\vec{B} = \vec{B}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

then $\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$

From Maxwell eqns. (Appendix) or constitutive equations
 $\mu_0 \vec{H} = \vec{B}$, where we assume $\mu = 1$ is appropriate.

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -E_s & 0 \end{vmatrix}$$

where $\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial x}$, $\frac{\partial}{\partial y} \equiv \frac{\partial}{\partial y}$, $\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial z}$, $E_s = E_s(\vec{r}, t)$

$$\vec{\nabla} \times \vec{E} = \hat{x} \frac{\partial}{\partial z} E_s - \hat{y} \cdot 0 + \hat{z} \left(-\frac{\partial}{\partial x} E_s \right)$$

$$= \hat{x} i k_z E_0 - \hat{z} i k_x E_s$$

where we have used that $\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$

hence $i k_z E_s = -i\omega B_{xp} = -i\omega \mu_0 H_{xp}$

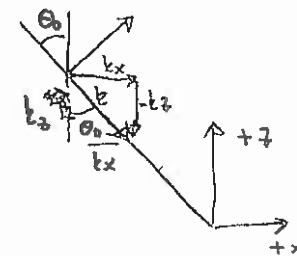
$$-i k_x E_s = -i\omega B_{zp} = -i\omega \mu_0 H_{zp}$$

$$0 = -i\omega B_{yp}$$

Hence $H_{xp} = \frac{1}{\mu_0} \frac{k_z}{\omega} E_s$

$$H_{zp} = \frac{1}{\mu_0} \frac{k_x}{\omega} E_s$$

$$H_{yp} = 0$$



$$\sin \theta_0 = \frac{k_x}{k}$$

$$\cos \theta_0 = -\frac{k_z}{k}$$

$$k_x = k \sin \theta_0$$

$$k_z = -k \cos \theta_0$$

$$\Rightarrow H_{xp} = \frac{1}{\mu_0} \frac{k}{\omega} \sin \theta_0 E_s$$

$$H_{yp} = 0$$

$$H_{zp} = \frac{1}{\mu_0} \frac{k}{\omega} \cos \theta_0 E_s \quad i(\vec{k} \cdot \vec{r} - \omega t)$$

$$\vec{H}_p = \frac{1}{\mu_0} \frac{k}{\omega} \left[\sin \theta_0 \hat{x} + \cos \theta_0 \hat{z} \right] e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

also note that

$$|\vec{H}_p| = H_{op} = \frac{1}{\mu_0} \frac{1}{v} E_0 \hat{s}$$

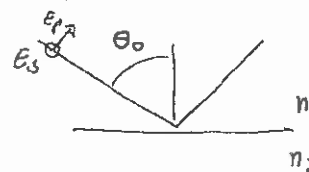
and that $\vec{H}_p(\vec{r}, t) = \frac{1}{\mu_0} \frac{k}{\omega} (\hat{s} \times \vec{E}_s(\vec{r}, t))$
 or $\vec{B}_p = \frac{1}{v} (\hat{s} \times \vec{E}_s)$

where $\hat{s} = \frac{\vec{k}}{k} = (n \sin \theta_0, 0, -\cos \theta_0)$

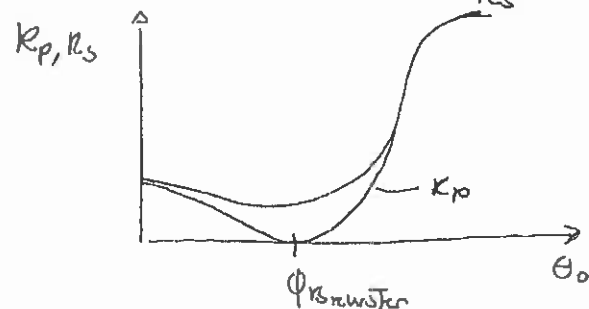
i.e. $(\vec{H}_p \perp \vec{E}_s \perp \vec{k})$

131b)

i) Polarisation by reflection from a plane interface surface; at the Brewster angle.



e.g. Air
water



at $\theta_0 = \phi_{\text{Brewster}}$, then $|r_p|^2 = R_p = 0$

hence only s-polarised light is reflected, i.e.

linearly polarised light

$$\tan(\phi_{\text{Brewster}}) = \frac{n_1}{n_2}$$

~~scribble~~

ii) Oriented polymer chains with an absorption that depends on the orientation of the electric field.

Also commonly known as "dichroic" polarizers:

Ames filter, sunglasses etc.

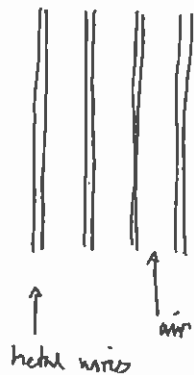


e.g.

(absorb) light when E-field $\parallel$ to chain

transmit light when E-field $\perp$ to chain

iii) Metallic wire-grid polarizers



absorb light when E-field $\parallel$ to wires (conduction electrons oscillates with field and acts as metallic reflector)
transmit light when E-field $\perp$ to wires

iv) Prism-like like polarizers composed of isotropic and/or anisotropic medium with different crystal orientation. Also known as birefringent polarizers (Hecht)

E.g. Glan Thompson polarizer

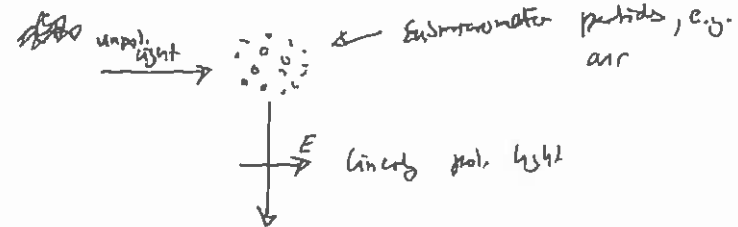
Rochon Polarizer

Wollaston prism

See Pedrotti page 362-363, Figure

15-15; 15-16. See Hecht page 344, 345

v) Polarization by scattering from particles



This is why you get better contrast looking at the sky with polarized "sunglasses" (sunglasses)

4)

retardance

$$\delta = \delta_e - \delta_o$$

$$= \frac{2\pi}{\lambda} (n_e - n_o) d$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$$

$$\delta = \pi \Rightarrow \text{half wave plate}$$

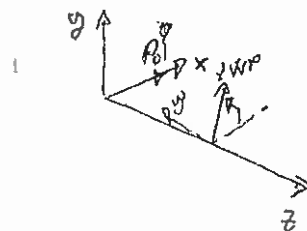
$$\delta = \frac{\pi}{2} \Rightarrow \text{quarter wave plate}$$

$$\delta_{HW} = \pi = \frac{2\pi}{\lambda} (n_e - n_o) d$$

$$d = \frac{\lambda}{2(n_e - n_o)} = 3 \times 10^4 \text{ nm} = 30 \mu\text{m}$$

$$\delta_{QW} \quad d = \frac{\lambda}{4(n_e - n_o)} = 15 \mu\text{m}$$

B1. d)



$$T_{WP} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}$$

Output Jones vector in x', y' system

$$\vec{J}_{x', y'} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix} \vec{R}(45) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{J}_{x, y} = \vec{R}(-45) \vec{T}_{x', y'} \vec{R}(45) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From Matrix Appendix

$$R(45) = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$R(-45) = \begin{bmatrix} \frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$\delta = \frac{3\pi}{2}$, which is going to
 with a CW photo

$$e^{i\delta} = \cos \frac{3\pi}{2} + i \sin \left(\frac{3\pi}{2} \right)$$

$$= 0 - i$$


$$\vec{J}_{x,y}(\delta) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \frac{1}{2} \underbrace{\begin{bmatrix} 1 & e^{i\delta} \\ -1 & e^{i\delta} \end{bmatrix}}_{\begin{bmatrix} 1 + e^{i\delta} \\ -1 + e^{i\delta} \end{bmatrix}} =$$

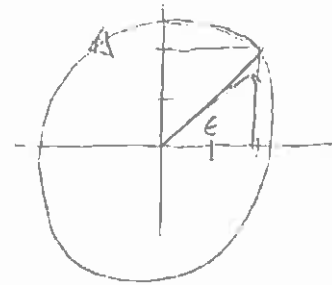
$$\vec{J}_{x,y}(\delta = \frac{3\pi}{2}) = \frac{1}{2} \begin{bmatrix} 1 + -i \\ -1 - i \end{bmatrix} = \frac{1}{2}(1-i) \begin{bmatrix} 1 \\ \frac{-1-i}{1-i} \end{bmatrix}$$

$$\frac{(-1-i)}{1-i} = \frac{(-1-i)(1+i)}{(1-i)(1+i)} = \frac{-1-i-i+1}{1^2+1^2}$$

$$= \frac{-2i}{2} = -i$$

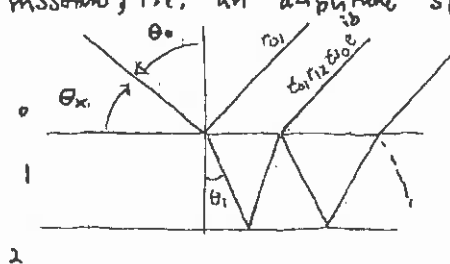
$$\vec{J}_{x,y} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \Rightarrow = \begin{bmatrix} 1 \\ i \tan \theta \end{bmatrix}$$

$$\tan \theta = \frac{E_{0,y}}{E_{0,x}} = -1$$



and the direction is
 the opposite, this is
 a left circular
 polarization state.

B3a) Analyze the reflection from a thin film on a substrate, i.e. an amplitude splitting interferometer.



The reflection coefficient can be written as

$$r_{012} = \frac{E_{\text{reflected field}}}{E_{\text{incident field}}}$$

$$= r_{01} + t_{01} t_{10} r_{12} e^{i\delta} + t_{01} t_{10} r_{12} r_{10} r_{12} e^{i2\delta} + \dots$$

$$(1) \quad = r_{01} + t_{01} t_{10} r_{12} e^{i\delta} \left[1 + \sum_{n=1}^{\infty} (r_{10} r_{12} e^{i\delta})^n \right]$$

where $\delta = \frac{4\pi}{\lambda} d_{\text{film}} n_{\text{film}} \cos \theta_{\text{film}}$

B3b) From the series of multiple reflections, it is observed from eqn. B3a (1) that for every

$$\delta = 0, 2\pi, 4\pi, \dots \quad r_{012} = r_{02} \quad , \quad \text{since}$$

$$d_{\text{film}} = 0 \Leftrightarrow \delta = 0 \quad \left(e^{i \frac{4\pi}{\lambda} n_{\text{film}} \cos \theta_{\text{film}} d_{\text{film}}} = e^{i\delta} \right)$$

As an approximation, we will estimate the thickness of the film between two maxima

$$\delta_{i+1} - \delta_i = 2\pi \quad , \quad \text{corresponding to}$$

$$\frac{4\pi d_{\text{film}}}{\lambda} n_{\text{film}} (\cos \theta_{\text{film}, i+1} - \cos \theta_{\text{film}, i}) = 2\pi$$

now $\cos(90 - \theta_i) = \sin \theta_x$, while

$$\cos \theta_{\text{film}} = \left(1 - \frac{n_0^2 \sin^2 \theta_0}{n_f^2} \right)^{\frac{1}{2}} \approx \left(1 - \frac{n_0^2 \cos^2 \theta_x}{n_f^2} \right)^{\frac{1}{2}}$$

use small's law $n_0 \sin \theta_0 = n_f \sin \theta_f$

$$\left(1 - \frac{n_0^2 \sin^2 \theta_0}{n_f^2} \right)^{\frac{1}{2}} = \cos \theta_{\text{film}}$$

$$\cos \theta_{\text{film}} \approx \left(1 - \frac{1}{n_f^2} (1 - \cos^2 \theta_x) \right)^{\frac{1}{2}} \approx \left(1 - \frac{1}{n_f^2} (1 - 2\cos^2 \theta_x) \right)^{\frac{1}{2}}$$

$$\approx \sqrt{\frac{n_f^2 - 1}{n_f^2}} (1 - 2\cos^2 \theta_x)^{\frac{1}{2}} \approx \sqrt{\frac{n_f^2 - 1}{n_f^2}} (1 - \cos \theta_x)$$

$$(\cos \theta_{f, i+1} - \cos \theta_{f, i}) \approx \cos \theta_{x, i} - \cos \theta_{x, i+1} \quad (\text{break only signs here})$$

$$4\pi \frac{d_{film}}{\lambda} n_{film} (\theta_{x_{i+1}} - \theta_{x_i}) \approx 2\pi$$

$$d_{film} \approx \frac{\lambda}{4n_{film}} \cdot \frac{1}{(\theta_{x_{i+1}} - \theta_{x_i})}$$

$$d_{film} \approx \frac{1.5406 \text{ \AA}}{2 \cdot (1 - 1.13 \times 10^{-5}) (4.8 - 3.9) \left(\frac{\pi}{180}\right)}$$

$$\approx 49 \text{ \AA}$$

~~What is~~

More exact solution can be found from adding into:

$$d_{film} = \frac{\lambda}{2 n_{film} \cdot \left[\left(1 - \frac{1}{n_{film}^2} \cos^2 \theta_{x_i}\right)^{\frac{1}{2}} - \left(1 - \frac{1}{n_{film}^2} \cos^2 \theta_{x_{i+1}}\right)^{\frac{1}{2}} \right]}$$

$$= \frac{\lambda}{2 \left[(n_f^2 - \cos^2 \theta_{x_i})^{\frac{1}{2}} - (n_f^2 - \cos^2 \theta_{x_{i+1}})^{\frac{1}{2}} \right]}$$

B3c) We observe that $n_{film} < n_{air}$
 (note that absorption of the order $k \sim 10^{-7}$ has been neglected)

$\Rightarrow$ total internal reflection in air

This happens for

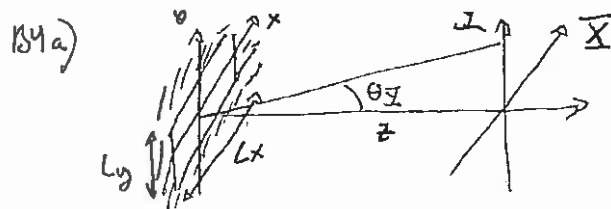
$$n_{air} \sin \theta_{air} = n_f \sin \theta_f \quad \theta_f = 90^\circ$$

$$\sin \theta_{air}^{critical} = \frac{n_f}{n_{air}}$$

$$\theta_{air}^{critical} = \sin^{-1} \left(\frac{n_f}{n_{air}} \right) \approx \frac{n_f}{n_{air}}$$

$$\theta_x = 90 - \frac{n_f}{n_{air}} \cdot \frac{\pi}{2}$$

$\theta_x^{critical/cut-off} = 0.2724^\circ$, in good agreement with the figure.



Fraunhofer diffraction integral is given by the appendix to 2:

$$1) \quad U(\bar{X}, \bar{Y}, z) = \frac{U_0^-(x, y, 0)}{i\lambda z} e^{ikz} e^{ik\frac{(\bar{X}^2 + \bar{Y}^2)}{2z}} \iint_{-\infty}^{\infty} t(x, y) e^{-i(k_{\bar{X}}x + k_{\bar{Y}}y)} dx dy$$

where we have introduced the transmission function for the aperture $t(x, y)$ defined by:

$$t(x, y) = \frac{U_0^+(x, y, 0)}{U_0^-(x, y, 0)} \quad (2)$$

where $U_0^-(x, y, 0)$ is the field amplitude incident on the aperture,

$$(3) \quad t(x, y) = \text{rect}\left(\frac{x}{L_x}, \frac{y}{L_y}\right) = \begin{cases} 1 & |x| < \frac{L_x}{2} \text{ and } |y| < \frac{L_y}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Eqn. (1) may be observed to be a Fourier transform in a particular, giving

$$U(\bar{X}, \bar{Y}, z) = \frac{U_0^-(x, y, 0)}{i\lambda z} e^{ikz} e^{ik\frac{(\bar{X}^2 + \bar{Y}^2)}{2z}} \mathcal{F}\{t(x, y)\}$$

where $\mathcal{F}\{t(x, y)\} = \mathcal{F}\left\{\text{rect}\left(\frac{x}{L_x}, \frac{y}{L_y}\right)\right\}$

$$\text{rect}\left(\frac{x}{L_x}, \frac{y}{L_y}\right) = \text{rect}\left(\frac{x}{L_x}\right) \cdot \text{rect}\left(\frac{y}{L_y}\right), \text{ (separable function)}$$

From the appendix we find

$$\mathcal{F}\left\{\text{rect}\left(\frac{x}{L_x}\right)\right\} = L_x \text{sinc}\left(\frac{k_{\bar{X}} L_x}{2}\right)$$

recall definition of Fourier Transform:

$$\mathcal{F}\left\{\text{rect}\left(\frac{x}{L_x}\right)\right\} = \int_{-\infty}^{\infty} \text{rect}\left(\frac{x}{L_x}\right) e^{-ik_{\bar{X}}x} dx$$

Fourier space variable

The Fourier space variables $k_{\bar{X}} \approx \frac{k\bar{X}}{z}$, $k_{\bar{Y}} \approx \frac{k\bar{Y}}{z}$, generally.

Hence

$$\mathcal{F}\{t(x, y)\} = L_x L_y \text{sinc}\left(\frac{k_{\bar{X}} L_x}{2}\right) \text{sinc}\left(\frac{k_{\bar{Y}} L_y}{2}\right)$$

The intensity, or irradiance, is given by:

$$\langle I \rangle = \frac{1}{2} \epsilon_0 c |U(\bar{X}, \bar{Y}, z)|^2 = \frac{1}{2} \epsilon_0 c U(\bar{X}, \bar{Y}, z) \cdot U(\bar{X}, \bar{Y}, z)^*$$

where $*$ denotes the complex conjugate.

Hence the intensity is given as

$$\langle I \rangle(x, y, z) = \frac{I_0}{\lambda^2 z^2} L_x^2 L_y^2 \text{sinc}^2\left(\frac{kx L_x}{2}\right) \text{sinc}^2\left(\frac{ky L_y}{2}\right),$$

where $I_0 = (U_0(x, y, 0))^2 \cdot \frac{1}{2} \epsilon_0 c$

B4b) The validity of the Fraunhofer approximation, with respect to the Fresnel approximation, is easily seen by observing the difference between the two, see Appendix. We see that if

(+) $e^{\frac{i k}{2z}(x^2+y^2)} \rightarrow 1$, then the Fraunhofer approximation holds.

The maximum of (+) is for $\frac{k}{2z}(L_x^2 + L_y^2)$

We thus require that

$$\frac{k}{2z}(L_x^2 + L_y^2) \ll 1$$

i.e.

$$\frac{\pi}{\lambda}(L_x^2 + L_y^2) \ll 1$$

$$= \frac{\pi \left((20 \times 10^3)^2 + (40 \times 10^3)^2 \right)}{633 \cdot 2 \times 10^9} \frac{[\text{nm}^2]}{[\text{nm}^2]}$$

$$= 0.005 \ll 1. \quad \text{Appears to be OK}$$

$$(e=c) \quad I(x, y, z) = \frac{I_0}{\lambda^2 z^2} \frac{\text{sinc}^2\left(\frac{(2N+1)kx L_x}{2}\right)}{\text{sinc}^2(kx L_x/2)} \frac{L_x^2 L_y^2 \text{sinc}^2(kx L_x/2)}{\text{sinc}^2(kx L_x/2)}$$

$$I(0, y, z) = I(0, 0, z) \cdot G_y \cdot \text{sinc}^2(ky L_y/2)$$

$$G_y \equiv \frac{\text{sinc}^2((2N+1)ky L_y/2)}{\text{sinc}^2(ky L_y/2)}$$

$$G_y \rightarrow (2N+1)^2 \text{ as } ky L_y = m\pi, \text{ where } m=0, \pm 1, \pm 2$$

$$ky L_y \approx \frac{2\pi}{\lambda} \frac{y}{z} L_y = m\pi$$

$$\Leftrightarrow y = \frac{z}{L_y} \left(\frac{m\lambda}{2} \right)$$

$$\text{sinc}^2(ky L_y/2) = 0 \quad \text{for } \frac{2\pi}{\lambda} \frac{y}{z} \frac{L_y}{2} = q\pi$$

$$\Leftrightarrow y = q \frac{\lambda z}{L_y}$$

see below.

