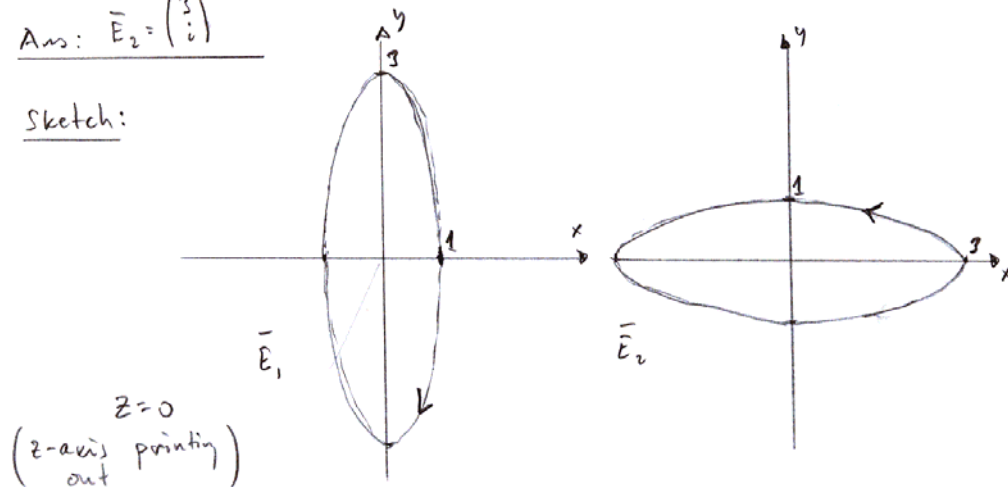


Optikk VK TFY4200
Solutions to examination June 3rd, 2004

P1: Ans: $\bar{E}_2 = \begin{pmatrix} 3 \\ i \end{pmatrix}$

Sketch:



Solution: $\bar{E}_1^* \cdot \bar{E}_2$ gives $1 \cdot e_{21} + (-3i) \cdot e_{22} = e_{21} + 3ie_{22} = 0$
 $\Rightarrow e_{21} = 3 ; e_{22} = i \Rightarrow \bar{E}_2 = \begin{pmatrix} 3 \\ i \end{pmatrix}$

Sketch pol. ellipse

$$\text{Def. } \bar{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_{0x} e^{i(kz - \omega t + \phi_x)} \\ E_{0y} e^{i(kz - \omega t + \phi_y)} \end{pmatrix} \bigg|_{\substack{\phi_x = \phi_y = 0 \\ z=0}} = \begin{pmatrix} E_{0x} e^{i(-\omega t)} \\ E_{0y} e^{i(-\omega t)} \end{pmatrix}$$

$$\bar{E}_1 = \begin{bmatrix} 1 \cdot e^{i(-\omega t)} \\ -3i \cdot e^{i(-\omega t)} \end{bmatrix} = \begin{bmatrix} 1 \cdot e^{i(-\omega t)} \\ 3 \cdot e^{i(-\omega t - \frac{\pi}{2})} \end{bmatrix} \bigg|_{\text{real part}} = \begin{pmatrix} 1 \cdot \cos(-\omega t) \\ 3 \cdot \cos(-\omega t - \frac{\pi}{2}) \end{pmatrix} \quad -\frac{\pi}{2} - \frac{\pi}{2}$$

in the same way

$$\bar{E}_2 = \begin{bmatrix} 3 \cdot \cos(-\omega t) \\ 1 \cdot \cos(-\omega t + \frac{\pi}{2}) \end{bmatrix}$$

Table		$\omega t = 0$	$= \frac{\pi}{4}$	$= \frac{\pi}{2}$	$= \frac{3\pi}{4}$
$\bar{E}_1$	x	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
	y	0	$-3/\sqrt{2}$	-3	$3/\sqrt{2}$
$\bar{E}_2$	x	3	$3/\sqrt{2}$	0	$-3/\sqrt{2}$
	y	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$

(P2)

Ans. $\frac{w_x}{w_y} \sim 0.44$ (0.4-0.5 range OK)

Detailed solution

At the focal plane the field distribution is the exact FT of the aperture field distribution.

$$U(x, y) = C \cdot \int_{-\frac{w_x}{2}}^{\frac{w_x}{2}} e^{-2\pi i \frac{x\xi}{\lambda z}} d\xi \int_{-\frac{w_y}{2}}^{\frac{w_y}{2}} e^{-2\pi i \frac{y\eta}{\lambda z}} d\eta$$

observation (FT) plane aperture plane

Integrals are separable. C complex constant.

Consider x -integral I_x : $\int_{-\frac{w_x}{2}}^{\frac{w_x}{2}} e^{-2\pi i \frac{x\xi}{\lambda z}} d\xi = \dots = w_x \cdot \frac{\sin \pi \left(\frac{w_x x}{\lambda z} \right)}{\pi \left(\frac{w_x x}{\lambda z} \right)}$

I_x has zeros for $\frac{w_x x}{\lambda z} = m \cdot 1$ where $m \geq 1$

Hence, distance between minima inversely proportional to w_x ($x = \frac{m \cdot \lambda \cdot z}{w_x}$). The same argument for vertical dimension (w_y & y).

$\frac{w_x}{w_y}$ can be directly measured in the figure.

E.g. locate the "seventh" minimum of each (x & y).

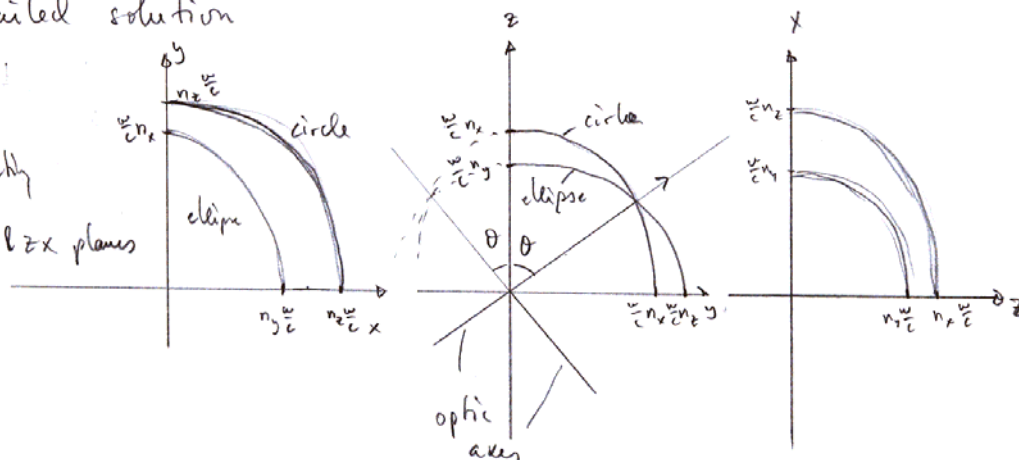
$$\frac{w_x}{w_y} \text{ is } \frac{\frac{1}{x} [\text{mm}]}{\frac{1}{y} [\text{mm}]} = \frac{19.5 \text{ mm}}{44 \text{ mm}} \approx 0.443$$

P3:

Ans. In the yz -plane $\pm 54^\circ$ from the z -axis.

Detailed solution

normal surface intersect the x, y, z planes



The normal surface(s) represent the solution(s) in terms of $\vec{k}$ or $\vec{e}$. It should be well known, easy to see that the optic axes are in the plane spanned by the extreme refractive indices (i.e. n_y & $n_z \rightarrow yz$ -plane). In this plane the ordinary ray (circle) is independent of θ $n = n_x$. The extraordinary ray follows the usual

$$\frac{1}{n_{eff}^2} = \frac{\cos^2 \theta}{n_y^2} + \frac{\sin^2 \theta}{n_z^2} \quad \text{with } \theta \text{ defined as above.}$$

Along optic axis extraordinary and ordinary ray propagate with the same speed. $\Rightarrow$

$$\frac{1}{n_x^2} = \frac{\cos^2 \theta}{n_y^2} + \frac{\sin^2 \theta}{n_z^2} ; \text{ a little algebra gives } \theta = 54^\circ$$

P4:

Ans.

alternative
answer

	A	B	C
$\epsilon_{ijk} \epsilon_{irs} a_r b_k c_j d_s$	$c_i a_j b_j d_j - c_k d_k b_i a_i$	$(\bar{c} \times \bar{b}) \cdot (\bar{a} \times \bar{d})$	$(\bar{c} \cdot \bar{a})(\bar{b} \cdot \bar{d}) - (\bar{c} \cdot \bar{d})(\bar{b} \cdot \bar{a})$
$\epsilon_{ijk} a_i b_j c_k - c_i \epsilon_{ijk} a_j b_k$	$a_i \epsilon_{ijk} b_j c_k - c_k \epsilon_{ijk} a_j b_i$	$\bar{a} \cdot (\bar{b} \times \bar{c}) - \bar{c} \cdot (\bar{a} \times \bar{b})$	0
$\epsilon_{ijk} \epsilon_{rsi} a_r b_j c_s$	$a_j b_i c_i - a_k b_j c_k$	$\bar{c} \times (\bar{a} \times \bar{b})$	$(\bar{c} \cdot \bar{b}) \bar{a} - (\bar{c} \cdot \bar{a}) \bar{b}$
— 11 —	3	3	2
	1	2	1
	2	1	3

Detailed solution.

row

- 1) First row trivial since only one combination have
4 $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ in all cases!

row

$$2) \epsilon_{ijk} a_i b_j c_k - c_i \epsilon_{ijk} a_j b_k = \epsilon_{ijk} a_i b_j c_k - \epsilon_{jki} a_j b_k c_i = \begin{matrix} \text{relabel indices} \\ \text{of 2nd term} \end{matrix} \begin{matrix} j \rightarrow i \\ k \rightarrow j \\ i \rightarrow k \end{matrix}$$

$\rightarrow \hat{L}$ permutable cyclic $\epsilon_{jki} = \epsilon_{ijk}$

$$= \epsilon_{ijk} a_i b_j c_k - \epsilon_{ijk} a_i b_j c_k = 0$$

With similar arguments is shown that $(A1) =$

$$a_i \epsilon_{ijk} b_j c_k - c_k \epsilon_{kij} a_i b_j = 0 \quad (C1)$$

It should be known that $\bar{a} \cdot (\bar{b} \times \bar{c}) = \bar{c} \cdot (\bar{a} \times \bar{b})$ (B2)

row

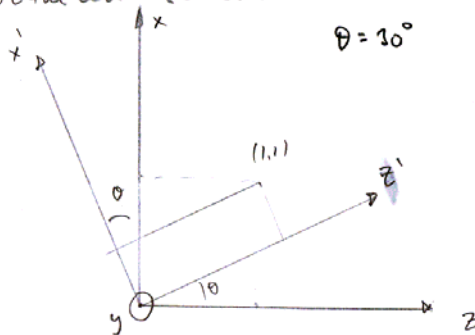
- 3) what remains should be equal - indeed it is!

PS:

Ans.

$$\begin{pmatrix} 2.3275 & 0 & -0.1342 \\ 0 & 2.25 & 0 \\ -0.1342 & 0 & 2.4825 \end{pmatrix}$$

Detailed solution



we have that

$$z' = \frac{\sqrt{2}}{2} z + \frac{1}{2} x$$

$$x' = -\frac{1}{2} z + \frac{\sqrt{2}}{2} x$$

The point (1,1) in the xz -system becomes

$$\left(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}-1}{2} \right) \text{ in the } x'z' \text{ system,}$$

seems OK.

Involving y the rotation matrix becomes

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

A 2nd rank tensor is transformed according to

$$a'_{ij} = l_{ip} l_{jq} a_{pq}$$

$$a'_{11} = l_{11} l_{11} a_{11} + l_{11} l_{12} a_{12} + l_{11} l_{13} a_{13} + l_{12} l_{11} a_{21} + l_{12} l_{12} a_{22} + l_{12} l_{13} a_{23} + l_{13} l_{11} a_{31} + l_{13} l_{12} a_{32} + l_{13} l_{13} a_{33}$$

and similar terms. Since a_{pq} diagonal and $l_{12} = l_{21} = l_{23} =$

$= l_{32} = 0$ it is easy to calculate a'_{ij} . Non-zero components

are: $a'_{11} = \frac{3}{4} a_{11} + \frac{1}{4} a_{33}$; $a'_{13} = \frac{\sqrt{2}}{4} a_{11} - \frac{\sqrt{2}}{4} a_{33}$; $a'_{22} = a_{22}$; $a'_{31} = \frac{\sqrt{2}}{4} a_{11} - \frac{\sqrt{2}}{4} a_{33}$;

$$a'_{33} = \frac{1}{4} a_{11} + \frac{3}{4} a_{33} \quad ; \text{ insert tensor values gives answer.}$$

P6:

Ans.

$$r_{\text{matrix}} = \begin{pmatrix} r_{11} & 0 & r_{13} \\ r_{21} & 0 & r_{23} \\ r_{31} & 0 & r_{33} \\ 0 & r_{42} & 0 \\ r_{51} & 0 & r_{53} \\ 0 & r_{62} & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ y_2 \\ x_2 \\ x_7 \end{matrix}$$

Detailed solution

follows from the definition of r_{matrix}
and $\tilde{y}_j$; see chapter 8 in
compendium.

8: Detailed solution

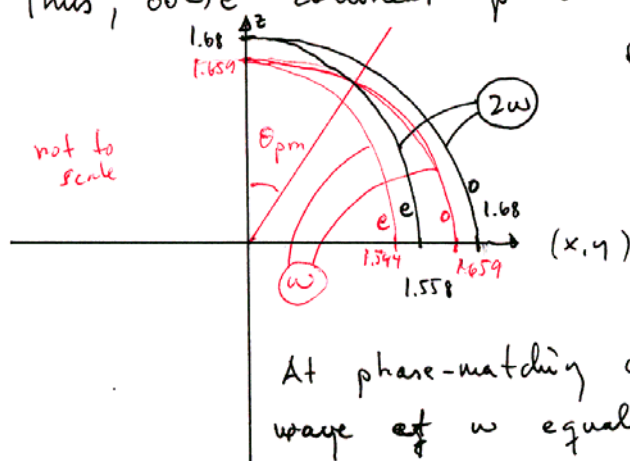
A: SHG process $\omega \rightarrow 2\omega$

$$\omega = \frac{2\pi c}{\lambda} \Rightarrow \text{to produce } \lambda_{2\omega} = 488 \text{ nm requires } \underline{\lambda_{\omega} = 976 \text{ nm}}$$

$$\text{from graph: } \lambda_{2\omega} \rightarrow n_o = 1.680 \quad n_e = 1.558 \quad (488 \text{ nm})$$

$$\lambda_{\omega} \rightarrow n_o = 1.659 \quad n_e = 1.544 \quad (976 \text{ nm})$$

Thus, $oo \rightarrow e$ collinear phase-matching is possible.



see graph

o-ordinary ray

e-extraordinary ray

At phase-matching angle (θ_{pm}) $\left. \begin{array}{l} \text{ref. index of} \\ \text{ordinary} \end{array} \right\}$ wave of ω equals n_e of e-wave @ 2ω

$$\Rightarrow \frac{1}{n_o^2} = \frac{\cos^2 \theta_{pm}}{n_o^2} + \frac{\sin^2 \theta_{pm}}{n_e^2}$$

$\uparrow$ $\quad \quad \quad \uparrow$ $\quad \quad \quad \uparrow$
 ω $\quad \quad \quad 2\omega$ $\quad \quad \quad 2\omega$

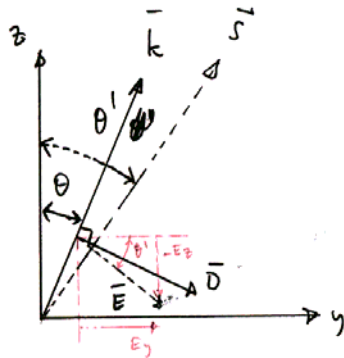
values:

$$\cos \theta_{pm} = \sqrt{\frac{\frac{1}{1.659^2} - \frac{1}{1.558^2}}{\frac{1}{1.680^2} - \frac{1}{1.558^2}}}$$

$$\Rightarrow \boxed{\begin{array}{l} \text{Ans.} \\ \theta_{pm} \approx 23.3^\circ \\ oo \rightarrow e \text{ SHG possible} \end{array}}$$

(77)

Take the coordinate system so that the optic axis is along z and the wave-vector $\vec{k}$ in the yz plane.



For the extraordinary wave the vector $\vec{B}$ (and $\vec{H}$) is perpendicular to the yz -plane.

The displacement field is always perpendicular to $\vec{k}$.

$$\text{Thus, } \begin{cases} D_y = D \cos \theta \\ D_z = -D \sin \theta \end{cases}$$

The components of the electric field are

$$E_y = \frac{D_y}{\epsilon_0 n_y^2} = \frac{D \cos \theta}{\epsilon_0 n_y^2} = \frac{D \cos \theta}{\epsilon_0 n_o^2}$$

$$E_z = \frac{D_z}{\epsilon_0 n_z^2} = \frac{-D \sin \theta}{\epsilon_0 n_z^2} = \frac{-D \sin \theta}{\epsilon_0 n_e^2}$$

The direction of the ray vector is proportional to the Poynting vector $\vec{P} \propto (\vec{E} \times \vec{B})$ (or $(\vec{E} \times \vec{H})$).

Thus, the ray is in the yz -plane, but not along $\vec{k}$. The angle between $\vec{P}$ and the optic axis (z)

$$\text{is then } -\frac{E_z}{E_y} = \tan \theta' = \frac{D \sin \theta \epsilon_0 n_o^2}{\epsilon_0 n_e^2 \cdot D \cos \theta} = \frac{n_o^2}{n_e^2} \tan \theta$$

answer

9. (10)

8:

B:

$$d\text{-matrix} = \begin{pmatrix} x & y & z & y & x & y \\ 0 & 0 & 0 & 0 & d_{15} & -d_{22} \\ -d_{22} & d_{22} & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix}$$

Kleinman symmetry:
we can permute
all x, y, z indices
freely.

$$\Rightarrow \left(\begin{array}{l} d_{15} : \text{stems from } \chi_{xyz} \\ d_{24} (= d_{15}) : \text{stems from } \chi_{yzy} \\ d_{31} : \text{stems from } \chi_{zxx} \\ d_{32} (= d_{31}) : \text{stems from } \chi_{zyy} \end{array} \right)_{\text{equal}} \Rightarrow \text{all equal} = d_{15} \text{ (or } d_{31})$$

$$o: (-\sin\phi, \cos\phi, 0)$$

$$e: (\cos\phi \cos\theta, \sin\phi \cos\theta, \sin\theta)$$

project to form d_{eff} for $oo \rightarrow e$ case (SHG)

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{15} & -d_{22} \\ -d_{22} & d_{22} & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x^2 \\ E_y^2 \\ E_z^2 \\ 2E_y E_z \\ 2E_x E_z \\ 2E_x E_y \end{pmatrix} = \begin{pmatrix} -2d_{22} E_x E_y \\ d_{22} (E_y^2 - E_x^2) \\ d_{31} (E_x^2 + E_y^2) \end{pmatrix}$$

2x o-wave
 $E_z\text{-comp} = 0$

$$= \begin{pmatrix} 2d_{22} \sin\phi \cos\phi \\ d_{22} (\cos^2\phi - \sin^2\phi) \\ + d_{31} \end{pmatrix} \text{ project onto } \begin{pmatrix} \cos\phi \cos\theta \\ \sin\phi \cos\theta \\ -\sin\theta \end{pmatrix}$$

$$= -d_{31} \sin\theta + \cos\theta d_{22} (3\sin\phi \cos^2\phi - \sin^3\phi)$$

$$= -d_{31} \sin\theta + d_{22} \cos\theta \sin 3\phi \leftarrow \text{Answer } d_{\text{eff}}(oo \rightarrow e)$$

other variants ok

$$\text{e.g. } 3\sin\phi - 4\sin^3\phi = \sin 3\phi$$

$$\theta = \theta_{\text{pm}} (\text{BA})$$

ϕ chosen for
optimum, e.g. $\phi = 30^\circ$

9:

start with $\frac{1}{\mu_0} \cdot \left(\frac{n}{c}\right)^2 \vec{s} \times \vec{s} \times \vec{E} = -\vec{D}$

use $\vec{D} = \epsilon_0 \cdot \tilde{K} \cdot \vec{E}$ & $\frac{1}{\epsilon_0 \mu_0} = c^2$; rewrite in tensor notation

Note

$$\Rightarrow n^2 \epsilon_{imn} \epsilon_{ijk} s_n s_j E_k = -K_{mq} \cdot E_q$$

some cleaning using $(\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj})$

$$\Rightarrow n^2 (E_m - s_m s_n \cdot E_n) = K_{mq} \cdot E_q$$

$\vec{s}$ arbitrary; K_{mq} taken as diagonal $\Rightarrow K_{mq} \cdot E_q = (n_m^2) E_m$
(i.e. $= n_x^2 E_x + n_y^2 E_y + n_z^2 E_z$)

$$\Rightarrow (n^2 - n_m^2) E_m = n^2 s_m \cdot s_n \cdot E_n$$

or

write out: $E_m = \frac{n^2 s_m s_n \cdot E_n}{(n^2 - n_m^2)}$

multiply with s_m on both sides $\Rightarrow$

$$s_m E_m = \frac{n^2 s_m^2 s_n \cdot E_n}{(n^2 - n_m^2)}$$

 $s_m E_m$ and $s_n E_n$ scalar product cancel

$$\Rightarrow \frac{1}{n^2} = \frac{s_m^2}{(n^2 - n_m^2)} = \frac{s_x^2}{(n^2 - n_x^2)} + \frac{s_y^2}{(n^2 - n_y^2)} + \frac{s_z^2}{(n^2 - n_z^2)}$$

Q.E.D.

see problem 6.15 in compendium!!