

Contact during the exam:
 Professor Arne Brataas
 Telephone: 73593647/90643170

Exam in TFY4205 Quantum Mechanics

December 4, 2009

09:00–13:00

Allowed help: Alternativ C

Approved calculator

K. Rottman: *Matematisk formelsamling*

Barnett and Cronin: *Mathematical formulae*

Some relations that might be useful are given at the end of this exam.

This problem set consists of 5 pages.

Problem 1. Time-dependent perturbation theory

Consider the initially unperturbed system described by the Hamiltonian $H_0(\vec{r})$, and the stationary, orthonormal eigenstates $\Psi_n^0(\vec{r}, t)$:

$$\Psi_n^0(\vec{r}, t) = \psi_n(\vec{r})e^{-iE_n t/\hbar}, \quad (1)$$

where

$$H_0(\vec{r})\psi_n(\vec{r}) = E_n\psi_n(\vec{r}). \quad (2)$$

We introduce the time-dependent perturbation $V(\vec{r}, t)$, so that the total Hamiltonian is

$$H(\vec{r}, t) = H_0(\vec{r}) + V(\vec{r}, t). \quad (3)$$

- a) We let $\Psi(\vec{r}, t)$ be eigenstates of the total Hamiltonian $H(\vec{r}, t)$, and expand them in terms of the known stationary states:

$$\Psi(\vec{r}, t) = \sum_k a_k(t) \Psi_k^0(\vec{r}, t). \quad (4)$$

What is the physical interpretation of the expansion coefficients $a_k(t)$?

In the rest of this problem, we will restrict ourselves to first-order time-dependent perturbation theory. If we assume that our unperturbed system was in the state described by $\Psi_i^0(\vec{r}, t)$ at $t \rightarrow -\infty$, one can show that

$$a_n(t) = \delta_{n,i} + \frac{1}{i\hbar} \int_{-\infty}^t dt' V_{ni}(t') e^{i\omega_{ni}t'}, \quad (5)$$

where

$$V_{ni}(t) = \int d\vec{r} (\psi_n(\vec{r}))^* V(\vec{r}, t) \psi_i(\vec{r}) = \langle n | V(\vec{r}, t) | i \rangle, \quad (6a)$$

and

$$\omega_{ni} = \frac{E_n - E_i}{\hbar}. \quad (6b)$$

- b) Consider an electron, moving in the x -direction, in a one dimensional harmonic oscillator potential:

$$H_0(x) = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2. \quad (7)$$

The electron is in the ground state at $t \rightarrow -\infty$. The electron is then subject to a time-dependent electric field $\mathcal{E}(t)$, so that the perturbation reads

$$V(x, t) = -e\mathcal{E}(t)x = e\mathcal{E}_0 x e^{-t^2/\tau^2}. \quad (8)$$

In which excited states is it possible to find the electron as $t \rightarrow +\infty$?

- c) Show that the probability P of finding the electron in an excited state as $t \rightarrow +\infty$ can be written

$$P = \frac{\pi e^2 \mathcal{E}_0^2 \tau^2}{2m\hbar\omega} \exp\left(-\frac{\omega^2 \tau^2}{2}\right). \quad (9)$$

You might find the following integral useful:

$$\int_{-\infty}^{\infty} dt \exp\left(-\frac{t^2}{\tau^2} + i\omega t\right) = \tau\sqrt{\pi} \exp\left[-\left(\frac{\omega\tau}{2}\right)^2\right].$$

- d) How should we choose τ in order to maximize the transition probability? Call the maximum transition probability $P_{\max}$, and derive an expression for $P_{\max}$.
- e) What happens to $P_{\max}$ when $\mathcal{E}_0$, the amplitude of the electric field, is increased towards $+\infty$? Derive an expression describing the validity of $P_{\max}$.
- (*Comment:* If you did not find an expression for $P_{\max}$ in 1 d), you can solve this problem by instead using P from Eq. (9), with τ as a positive constant.)

Problem 2. Scattering theory

In this problem we will consider a three dimensional stationary scattering problem, described by the stationary Schrödinger equation

$$(\nabla^2 + k^2) \psi(\vec{r}) = U(\vec{r})\psi(\vec{r}), \quad (10)$$

where $k = \sqrt{2mE/\hbar^2}$ and $U(\vec{r}) = 2mV(\vec{r})/\hbar^2$. This equation describes a particle of mass m and energy E that scatters at the potential $V(\vec{r})$, that we take to be at rest at the origin. At large (asymptotic) distances, the wave function of the particle is

$$\psi(\vec{r}) \simeq e^{i\vec{k}\cdot\vec{r}} + f(\vartheta, \varphi) \frac{e^{ikr}}{r}, \quad (11)$$

where $f(\vartheta, \varphi)$ is the scattering amplitude.

- a) Give a physical definition of the differential and the total scattering cross section, and write down how these quantities are related to the scattering amplitude $f(\vartheta, \varphi)$ (no derivations are required).

In the first Born approximation, the scattering amplitude is

$$f^B(\vartheta, \varphi) = -\frac{1}{4\pi} \int d\vec{r}' e^{-i\vec{q}\cdot\vec{r}'} U(\vec{r}'), \quad (12)$$

where $\vec{q} = \vec{k}' - \vec{k} = k\vec{r}/r - \vec{k}$.

b) Consider the spherically symmetric potential described by

$$V_S(r) = \frac{V_0 e^{-\lambda r}}{\lambda r}, \quad (13)$$

where λ^{-1} characterizes the range of the potential. Use the first Born approximation to find an expression for the scattering amplitude, and show that the differential scattering cross section for this potential can be written

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\lambda\hbar^2} \right)^2 \frac{1}{(\lambda^2 + q^2)^2}, \quad (14)$$

where $q = 2k \sin \theta/2$.

c) The Coulomb potential is

$$V_C(r) = \frac{ZZ'e^2}{4\pi\epsilon_0 r}. \quad (15)$$

Use the result from **b)** to find the differential scattering cross section for the potential V_C .

d) The Born approximation is valid if the following inequality holds:

$$\int_0^\infty dr' r' |U(r')| \ll 1. \quad (16)$$

The potential V_S is strong enough to form a bound state if

$$\frac{2m|V_0|}{\lambda^2\hbar^2} \geq 2.7. \quad (17)$$

Discuss the validity of the Born approximation for the potential V_S based on the requirement in Eq. (16) and the condition in Eq. (17)! Is the first Born approximation valid for the potential V_C ?

Problem 3. Quantization of the Electromagnetic Fields

The Hamiltonian for the electromagnetic field in vacuum is

$$H = \frac{1}{2} \int d^3r (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}). \quad (18)$$

We choose the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, where $\mathbf{A}$ is the electromagnetic vector potential. The electromagnetic fields can be expressed in terms of the electromagnetic vector potential as

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A}, \\ \mathbf{H} &= \mathbf{B}/\mu_0, \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t}, \\ \mathbf{D} &= \epsilon_0 \mathbf{E}, \end{aligned}$$

where ε_0 is the dielectricity constant and μ_0 is the magnetic permeability that are related by the velocity of light $c^2 = (\mu_0 \varepsilon_0)^{-1}$. The Hamiltonian for the electromagnetic field can then be expressed in terms of the electromagnetic vector potential as

$$H = \frac{\varepsilon_0 c^2}{2} \int d^3r \left[\left(\frac{\partial \mathbf{A}}{\partial ct} \right)^2 + (\nabla \times \mathbf{A})^2 \right].$$

The electromagnetic field can be quantized and expressed as

$$\hat{\mathbf{A}}(r, t) = \sum_{\mathbf{k}\lambda} \mathbf{e}_{\mathbf{k}\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V c k}} \left[a_{\mathbf{k}\lambda} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} + a_{\mathbf{k}\lambda}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \right], \quad (19)$$

where λ denotes the two polarization directions ($\lambda = 1$ or $\lambda = 2$), $\mathbf{e}_{\mathbf{k},\lambda}$ is the polarization vector, and $\mathbf{k}$ is the wavevector. The operator $a_{\mathbf{k},\lambda}$ satisfies

$$\left[a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}^\dagger \right] = \delta_{\mathbf{k}\lambda, \mathbf{k}'\lambda'}.$$

The polarization vectors satisfy

$$\begin{aligned} \mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{e}_{\mathbf{k}\lambda'} &= \delta_{\lambda\lambda'}, \\ \mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{k} &= 0, \\ \mathbf{e}_{\mathbf{k}1} \cdot \mathbf{e}_{\mathbf{k}1} &= 1, \\ \mathbf{e}_{\mathbf{k}2} \cdot \mathbf{e}_{\mathbf{k}2} &= -1. \end{aligned}$$

- a) Using the expression for the electromagnetic vector potential (19), the Hamiltonian can be written as

$$H = \sum_{\mathbf{k}\lambda} \hbar \omega_{\mathbf{k}} \left(a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} + \frac{1}{2} \right),$$

where $\omega_{\mathbf{k}} = ck$. What are the physical interpretations of the quantities $\hbar \omega_{\mathbf{k}}$, $a_{\mathbf{k}\lambda}$, $a_{\mathbf{k}\lambda}^\dagger$, and $a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda}$?

- b) Explicitly demonstrate that the Hamiltonian (18) can be written as

$$H = \sum_{\mathbf{k}\lambda} \hbar \omega_{\mathbf{k}\lambda} \left(a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} + \frac{1}{2} \right).$$

Some potentially useful relations

Harmonic oscillator

The Hamiltonian of a one dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right), \quad (20)$$

where the ladder operators are defined as

$$a = \sqrt{\frac{m\omega}{2\hbar}} q + \frac{i}{\sqrt{2m\hbar\omega}} p, \quad \text{and} \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} q - \frac{i}{\sqrt{2m\hbar\omega}} p.$$

This is equivalent to

$$q = \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a), \quad \text{and} \quad p = i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a).$$

The ladder operators satisfy

$$[a, a^\dagger] = 1,$$

and

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle, \\ a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \end{aligned}$$

where $|n\rangle$ are the orthonormalized eigenstates of H in Eq. (20):

$$H|n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle = E_n |n\rangle.$$

Vector algebra

For the vectors **A**, **B**, **C**, and **D**, this holds

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).$$