

Final exam in

TFY 4/205

Quantum mechanics II

Fall 2013

Solutions:

Problem 1

$$\begin{aligned} a) \quad \underline{[a, a^\dagger]} &= \left[i\sqrt{\frac{1}{2m\hbar\omega}} p + \sqrt{\frac{m\omega}{2\hbar}} x, \right. \\ &\quad \left. -i\sqrt{\frac{1}{2m\hbar\omega}} p + \sqrt{\frac{m\omega}{2\hbar}} x \right] \\ &= i\frac{1}{2\hbar} [p, x] - i\frac{1}{2\hbar} [x, p] \\ &= -i\frac{1}{2\hbar} i\hbar - i\frac{1}{2\hbar} i\hbar = \underline{\underline{1}} \end{aligned}$$

$$\begin{aligned} b) \quad \left. \begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) \\ p &= i\sqrt{\frac{\hbar m\omega}{2}} (a^\dagger - a) \end{aligned} \right\} \end{aligned}$$

$$\underline{H_0 = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} x^2}$$

$$= -\frac{1}{2m} \frac{\hbar m\omega}{2} (a^+ - a)(a^+ - a)$$

$$+ \frac{m\omega^2}{2} \frac{\hbar}{2m\omega} (a^+ + a)(a^+ + a)$$

$$= \frac{\hbar\omega}{2} (a^+ a + a a^+) = \underline{\hbar\omega(a^+ a + \frac{1}{2})}$$

$$\text{Let } a^+ a |n\rangle = n |n\rangle.$$

$$\text{Then } H_0 |n\rangle = \underbrace{\hbar\omega(n + \frac{1}{2})}_{E_n} |n\rangle$$

We must find n .

$$\text{We have } \underline{[a^+ a, a]} = \underbrace{a^+ a a}_{a a^+ - 1} - a a^+ a = \underline{-a}$$

Hence,

$$\begin{aligned} a^+ a (a |n\rangle) &= a(a^+ a |n\rangle) - a |n\rangle \\ &= (n-1) a |n\rangle \end{aligned}$$

In general:

$$a^\dagger a (a^k |n\rangle) = (n-k) a^k |n\rangle$$

$\Rightarrow$ $a^k |n\rangle$ is an eigenvector of $a^\dagger a$ with eigenvalue $(n-k)$.

$\uparrow$
This is an integer.

The norm is always positive:

$$(\langle n|a^\dagger)(a|n\rangle) = n \langle n|n\rangle \geq 0 \Rightarrow$$

$$\underline{(n-k) \geq 0} \Rightarrow \begin{array}{l} 1. \underline{n \text{ is an integer.}} \\ 2. \underline{a|n=0\rangle = 0|n=0\rangle} \end{array}$$

Thus,

$$\underline{H_0 |n\rangle = \hbar \omega \left(n + \frac{1}{2}\right) |n\rangle}$$

where $n = 0, 1, 2, 3, \dots$

c) In the following, we assume $t=0$.

4

$$\text{We assume } a|m\rangle = c|m-1\rangle$$

$$\text{where } \langle m|m\rangle = \langle m-1|m-1\rangle = 1.$$

$$\langle m|a^\dagger a|m\rangle = m \langle m|m\rangle = m$$

$$\langle m|a^\dagger a|m\rangle = (\langle m|a^\dagger)(a|m\rangle) = c^2 \langle m-1|m-1\rangle = c^2$$

$$\Rightarrow \underline{c = \sqrt{m}}$$

$$\langle m|m\rangle = \delta_{m,m}$$

$\Rightarrow$

$$\underline{\langle m|a|m\rangle = \sqrt{m} \langle m|m-1\rangle = \sqrt{m} \delta_{m,m-1}}$$

$\downarrow$

$$\underline{\langle m|a^\dagger|m\rangle = \langle m|a|m\rangle^\dagger = \sqrt{m} \delta_{m,m-1}}$$

$$= \underline{\sqrt{m+1} \delta_{m,m+1}}$$

5

$$\begin{aligned}
 d) \quad \underline{\underline{\langle m | x | m \rangle}} &= \sqrt{\frac{\hbar}{2m\omega}} \left\{ \langle m | a^\dagger | m \rangle + \langle m | a | m \rangle \right\} \\
 &= \sqrt{\frac{\hbar}{2m\omega}} \left\{ \sqrt{m+1} \delta_{m, m+1} + \sqrt{m} \delta_{m, m-1} \right\}
 \end{aligned}$$

$$\begin{aligned}
 \underline{\underline{\langle m | p | m \rangle}} &= i \sqrt{\frac{\hbar m \omega}{2}} \left\{ \langle m | a^\dagger | m \rangle - \langle m | a | m \rangle \right\} \\
 &= i \sqrt{\frac{\hbar m \omega}{2}} \left\{ \sqrt{m+1} \delta_{m, m+1} - \sqrt{m} \delta_{m, m-1} \right\}
 \end{aligned}$$

$$\begin{aligned}
 e) \quad \underline{\underline{\langle m | x^2 | m \rangle}} &= \sum_{k=0}^{\infty} \langle m | x | k \rangle \langle k | x | m \rangle \\
 &= \sum_{k=0}^{\infty} \frac{\hbar}{2m\omega} \left\{ \sqrt{k+1} \delta_{m, k+1} + \sqrt{k} \delta_{m, k-1} \right\} \\
 &\quad \left\{ \sqrt{m+1} \delta_{k, m+1} + \sqrt{m} \delta_{k, m-1} \right\} \\
 &= \frac{\hbar}{2m\omega} \left\{ \sqrt{(m+2)(m+1)} \delta_{m, m+2} \right. \\
 &\quad \left. + (2m+1) \delta_{m, m} + \sqrt{m(m-1)} \delta_{m, m-2} \right\}
 \end{aligned}$$

$$\underline{\underline{\langle m | p^2 | m \rangle = \sum_{k=0}^{\infty} \langle m | p | k \rangle \langle k | p | m \rangle}}$$

$$= - \frac{\hbar m \omega}{2} \sum_{k=0}^{\infty} \left\{ \sqrt{k+1} \delta_{m, k+1} - \sqrt{k} \delta_{m, k-1} \right\}$$

$$\left\{ \sqrt{m+1} \delta_{k, m+1} - \sqrt{m} \delta_{k, m-1} \right\}$$

$$\underline{\underline{= - \frac{\hbar m \omega}{2} \left\{ \sqrt{(m+2)(m+1)} \delta_{m, m+2} \right. \\ \left. - (2m+1) \delta_{m, m} + \sqrt{m(m-1)} \delta_{m, m-2} \right\}}}}$$

$$f) \left. \begin{array}{l} \langle m | x | m \rangle = 0 \\ \langle m | p | m \rangle = 0 \end{array} \right\} \underline{\underline{\langle x \rangle^2 = \langle p \rangle^2 = 0}}$$

$$\langle m | x^2 | m \rangle = \frac{\hbar}{m \omega} \left(m + \frac{1}{2} \right) \Rightarrow \underline{\underline{\Delta x = \sqrt{\frac{\hbar}{m \omega} \left(m + \frac{1}{2} \right)}}}$$

$$\langle m | p^2 | m \rangle = \hbar m \omega \left(m + \frac{1}{2} \right) \Rightarrow \underline{\underline{\Delta p = \sqrt{\hbar m \omega \left(m + \frac{1}{2} \right)}}}$$

$$\underline{\underline{\Delta x \Delta p = \hbar \left(m + \frac{1}{2} \right)}}$$

$$g) \langle H_0 \rangle = \hbar \omega (n + \frac{1}{2})$$

7

$$\langle H_0 \rangle = \frac{1}{2m} \langle p^2 \rangle + \frac{m\omega^2}{2} \langle x^2 \rangle$$

$$\left. \begin{aligned} \frac{1}{2m} \langle p^2 \rangle &= \frac{1}{2} \hbar \omega (n + \frac{1}{2}) \\ \frac{m\omega^2}{2} \langle x^2 \rangle &= \frac{1}{2} \hbar \omega (n + \frac{1}{2}) \end{aligned} \right\}$$

We see that the expectation value of the kinetic and potential energies each are one half the expectation value of the total energy.

Problem 2

$$\begin{aligned} a) H_a &= H_0 + \alpha_1 x = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 + \alpha_1 x \\ &= \frac{p^2}{2m} + \frac{m\omega^2}{2} \left(x + \frac{\alpha_1}{m\omega^2} \right)^2 - \frac{\alpha_1^2}{2m\omega^2} \end{aligned}$$

$$\Rightarrow H_a |n'\rangle = E_{n'}^a |n'\rangle$$

$$\underline{\underline{E_{n'}^a = \hbar\omega \left(n' + \frac{1}{2}\right) - \frac{\alpha_1^2}{2m\omega^2}}}$$

$$\underline{\underline{n' = 0, 1, 2, \dots}}$$

$$\begin{aligned} b) \quad H_r &= H_0 + \alpha_2 x^2 = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 + \alpha_2 x^2 \\ &= \frac{p^2}{2m} + \frac{m}{2} \left(\omega^2 + \frac{2\alpha_2}{m} \right) x^2 \end{aligned}$$

$$\Rightarrow H_b |n''\rangle = E_{n''}^b |n''\rangle$$

$$\underline{\underline{E_{n''}^b = \hbar \sqrt{\omega^2 + \frac{2\alpha_2}{m}} \left(n'' + \frac{1}{2}\right)}}$$

$$\underline{\underline{n'' = 0, 1, 2, \dots}}$$

$$\begin{aligned} c) \quad \underline{\underline{|n'\rangle}} &= \sum_{m=0}^{\infty} |m\rangle \langle m|n'\rangle = \underline{\underline{\sum_{m=0}^{\infty} \delta_{mn'} |m\rangle}} \\ \underline{\underline{|n''\rangle}} &= \sum_{m=0}^{\infty} |m\rangle \langle m|n''\rangle = \underline{\underline{\sum_{m=0}^{\infty} \delta_{mn''} |m\rangle}} \end{aligned}$$

9

d)

$$\underline{\underline{\langle x | n' \rangle = \left(\frac{m\omega}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{2^{n'} n'!}} H_{n'} \left(\sqrt{\frac{m\omega}{\hbar}} \left(x + \frac{\alpha_1}{m\omega} \right) \right) e^{-m\omega \left(x + \frac{\alpha_1}{m\omega} \right)^2 / 2\hbar}}}$$

e)

$$\underline{\underline{\langle x | n'' \rangle = \left(\frac{m\omega''}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{2^{n''} n''!}} H_{n''} \left(\sqrt{\frac{m\omega''}{\hbar}} x \right) e^{-m\omega'' x^2 / 2\hbar}}}$$

where $\underline{\underline{\omega'' = \sqrt{\omega^2 + \frac{2\alpha_1}{m}}}}$

f) We have from 1d that $\langle n | x | n \rangle = 0$.

Hence, we now have

$$\langle n' | \left(x + \frac{\alpha_1}{m\omega} \right) | n' \rangle = 0 \Rightarrow$$

$$\underline{\underline{\langle n' | x | n' \rangle = - \frac{\alpha_1}{m\omega^2} = \frac{qE}{m\omega^2}}}$$

where $\alpha_1 = -qE$

The average dipole moment is

10

therefore

$$\underline{\underline{\langle n' | q x | n' \rangle = \frac{q^2}{m\omega^2} E}}$$

Problem 3

a) We set

$$|n\rangle = |n^{(0)}\rangle$$

$$\langle n | H_1 | n \rangle = \alpha_1 \langle n | x | n \rangle = 0$$

$$\sum_{m \neq n} \frac{|\langle m | H_1 | n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{\alpha_1^2}{2m\omega^2} \left\{ \frac{(\sqrt{n+1})^2}{-1} + \frac{(\sqrt{n})^2}{1} \right\}$$

$$= \frac{\alpha_1^2}{2m\omega^2} (n - n - 1) = - \frac{\alpha_1^2}{2m\omega^2}$$

$\Rightarrow$

$$\underline{\underline{E_n = \hbar\omega(n + \frac{1}{2}) - \frac{\alpha_1^2}{2m\omega^2} \lambda^2 + O(\lambda^3)}}$$

Compare this with 2a.

u)

$$\underline{\langle m | H_1 | m \rangle} = \alpha_2 \frac{\hbar}{2m\omega} (2m+1)$$

$$= \alpha_2 \frac{\hbar}{m\omega} \left(m + \frac{1}{2}\right)$$

(From 1c).

$$\sum_{m \neq n} \frac{|\langle m | H_1 | n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \alpha_2^2 \left(\frac{\hbar}{2m\omega}\right)^2 \left\{ \frac{(n+2)(n+1)}{\hbar\omega(-2)} + \frac{n(n-1)}{\hbar\omega 2} \right\} = -\alpha_2^2 \frac{\hbar\omega}{2m^2\omega^2} \left(n + \frac{1}{2}\right)$$

Hence, we have

$$\underline{E_n} = \hbar\omega \left(n + \frac{1}{2}\right) + \lambda \alpha_2 \frac{\hbar\omega}{m\omega^2} \left(n + \frac{1}{2}\right) - \lambda^2 \alpha_2^2 \frac{\hbar\omega}{2(m\omega^2)^2} \left(n + \frac{1}{2}\right) + O(\lambda^3)$$

$$= \hbar\omega \left(n + \frac{1}{2}\right) \left(1 + \frac{\lambda \alpha_2}{m\omega^2} - \frac{1}{2} \left(\frac{\lambda \alpha_2}{m\omega^2}\right)^2 + O(\lambda^3)\right)$$

Compare this expression to the exact one in 2b: expand $\sqrt{\omega + \frac{\alpha_2}{m}}$ in powers of α_2 !

Problem 4

12

a) Use Cartesian coordinates:

$$\int_{-\infty}^{+\infty} dx e^{-\left(\frac{x}{a}\right)^2 - i(k_x - k_x')x} = a\sqrt{\pi} e^{-\frac{a^2}{4}(k_x - k_x')^2}$$

Then,

$$\begin{aligned} & \frac{\int d\vec{r} V(\vec{r}) e^{-i(\vec{k} - \vec{k}') \cdot \vec{r}}}{=} \pi^{3/2} \alpha e^{-\frac{a^2}{4}(k_x - k_x')^2 - \frac{b^2}{4}(k_y - k_y')^2 - \frac{c^2}{4}(k_z - k_z')^2} \\ & = \pi^{3/2} \alpha e^{-\frac{a^2}{4}q_x^2 - \frac{b^2}{4}q_y^2 - \frac{c^2}{4}q_z^2} \end{aligned}$$

Where we have introduced $\vec{q} = \vec{k} - \vec{k}'$

$$\left. \begin{aligned} q_x &= k \sin \theta \cos \phi \\ q_y &= k \sin \theta \sin \phi \\ q_z &= k(1 - \cos \theta) \end{aligned} \right\}$$

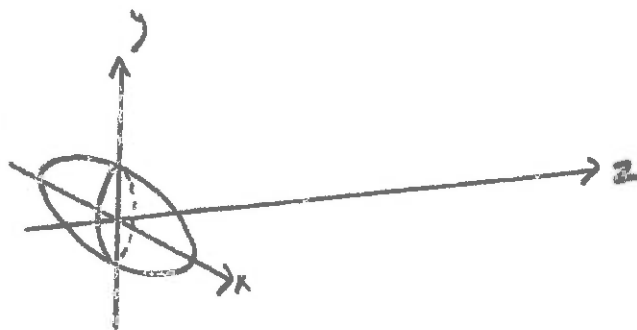
$$f^B = - \frac{m \alpha \sqrt{\pi}}{2 \hbar^2} e^{-\frac{k^2}{4} (a^2 \sin^2 \theta \cos^2 \phi + b^2 \sin^2 \theta \sin^2 \phi + c^2 (1 - \cos \theta)^2)} \quad 13$$

The differential cross section is

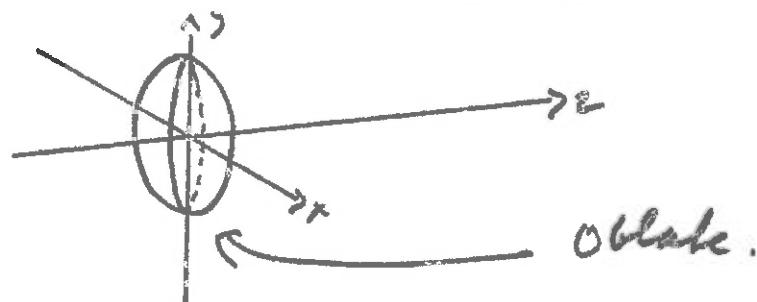
$$\frac{d\sigma}{d\Omega} = \left(\frac{m \alpha \sqrt{\pi}}{2 \hbar^2} \right)^2 e^{-\frac{k^2}{2} (a^2 \sin^2 \theta \cos^2 \phi + b^2 \sin^2 \theta \sin^2 \phi + c^2 (1 - \cos \theta)^2)}$$

b)

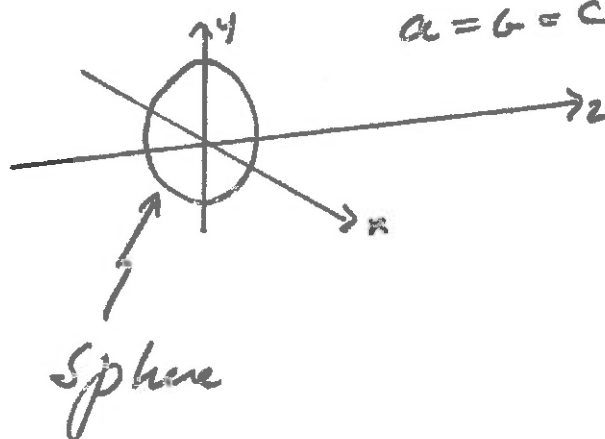
$$a > b > c$$



$$a = b > c$$



$$a = b = c$$



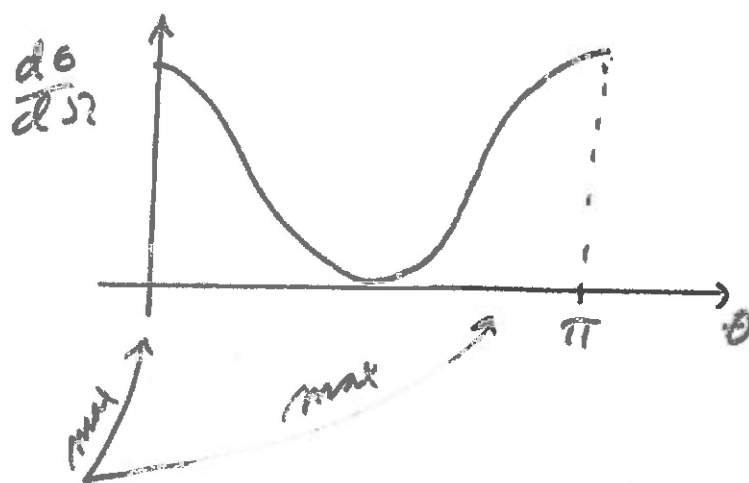
c) When $a > b > c$, the scattering has no symmetries.

When $a = b > c$, the ϕ -dependence disappears:

$$\frac{d\sigma}{d\Omega} = \left(\frac{m\alpha\sqrt{\pi}}{2k^2} \right)^2 e^{-\frac{k^2}{2}(a^2 \sin^2\theta + c^2(1-\cos\theta)^2)}$$

Suppose now $a \gg c$:

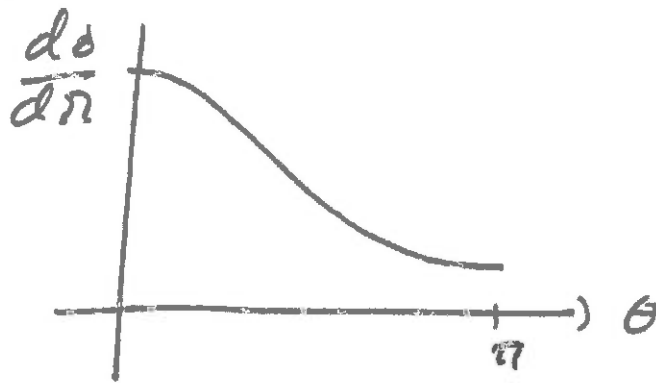
$$\frac{d\sigma}{d\Omega} \propto e^{-\frac{k^2}{2}a^2 \sin^2\theta}$$



This is characteristic of a punchable disk.

Suppose now $a = b = c$.

15



This is typical of scattering from a spherical repulsive potential with finite range.



Vurd.ordning: S 12.12.2013 kl 15:00 4t
Professor Alex Hansen

Karakterregel: Bokstavkarakterer

Kandnr	Resultat
--------	----------

10000	A
-------	---

10002	B
-------	---

10003	E
-------	---

10004	A
-------	---

10006	E
-------	---

10007	C
-------	---

10008	E
-------	---

10011	B
-------	---

10012	E
-------	---

10013	C
-------	---

10015	B
-------	---

10016	C
-------	---

10017	D
-------	---

10018	E
-------	---

10019	F
-------	---

10020	C
-------	---

10026	E
-------	---

10028	C
-------	---

10030	A
-------	---

Antall kandidater til sensur i TFY4205 S1 2013 12: 19

21/1/14

Dato

Alex Hansen

Sensorenes underskrifter