

## TFY 4205 Quantum Mechanics II

Final exam Dec. 10, 2015

Solution Set.Problem 1

$$a) \underline{[a, a^\dagger]} = \left[ \sqrt{\frac{m\omega}{2\hbar}} q + \frac{i}{\sqrt{2m\hbar\omega}} p, \sqrt{\frac{m\omega}{2\hbar}} q - \frac{i}{\sqrt{2m\hbar\omega}} p \right]$$

$$= -i \frac{1}{2\hbar} [q, p] + i \frac{1}{2\hbar} [p, q]$$

$$= -i \frac{1}{2\hbar} (+i\hbar) + i \frac{1}{2\hbar} (-i\hbar) = \underline{\underline{1}}$$

b) From (3) and (4)

$$\left. \begin{aligned} q &= \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) \\ p &= i \sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a) \end{aligned} \right\}$$

$$\underline{\underline{H}} = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} q^2$$

$$= -\frac{1}{2m} \frac{m\hbar\omega}{2} (a^+ - a)^2 + \frac{m\omega^2}{2} \frac{\hbar}{2m\omega} (a^+ + a)^2$$

$$= \frac{\hbar\omega}{4} \left\{ (a^+)^2 + a^+a + aa^+ + a^2 - (a^+)^2 \right. \\ \left. + a^+a + aa^+ - a^2 \right\}$$

$$= \frac{\hbar\omega}{2} \{ \underbrace{a^+a + aa^+}_{\substack{\uparrow \\ aa^+ = a^+a + 1}} \} = \underline{\underline{\hbar\omega (a^+a + \frac{1}{2})}}$$

c) We need

$$(a^+a)(a^+)^n = (a^+a)a^+(a^+)^{n-1}$$

$$= a^+ \underbrace{(aa^+)}_{[a, a^+] = 1} (a^+)^{n-1} = a^+ (a^+a + 1) (a^+)^{n-1}$$

$$= a^+ (a^+a)(a^+)^{n-1} + (a^+)^n$$

$$= a^+ [(a^+a)a^+] (a^+)^{n-2} + (a^+)^n$$

$$= (a^+)^2 (a^+a)(a^+)^{n-2} + 2(a^+)^n = \dots$$

$$= n(a^+)^n$$

↑  
Repeat...

Hence,

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$$\underline{H |n\rangle} = \frac{\hbar\omega}{\sqrt{n!}} e^{-i(n+\frac{1}{2})\omega t} (a^\dagger a + \frac{1}{2}) (a^\dagger)^n |0\rangle$$

$$= \frac{\hbar\omega}{\sqrt{n!}} e^{-i(n+\frac{1}{2})\omega t} (n + \frac{1}{2}) (a^\dagger)^n |0\rangle$$

$$= \underline{\hbar\omega(n + \frac{1}{2}) |n\rangle}$$

$$\underline{i\hbar \frac{d}{dt} |n\rangle} = \frac{i\hbar}{\sqrt{n!}} \left( \frac{d}{dt} e^{-i(n+\frac{1}{2})\omega t} \right) (a^\dagger)^n |0\rangle$$

$$= \underline{\hbar\omega(n + \frac{1}{2}) |n\rangle}$$

$\Rightarrow$  Left hand side = Right hand side.

d) We need  $a (a^\dagger)^n = (a a^\dagger) (a^\dagger)^{n-1}$

$$= (a^\dagger a + 1) (a^\dagger)^{n-1}$$

$$= a^\dagger (a a^\dagger)^{n-1} + (a^\dagger)^n$$

$$= \dots$$

$$= n (a^\dagger)^{n-1}$$

$$\text{Hence, } \underline{a|m\rangle} = \frac{e^{-i(m+\frac{1}{2})\omega t}}{\sqrt{m!}} a (a^\dagger)^m |0(t=0)\rangle \quad 4$$

$$= \frac{e^{-i(m+\frac{1}{2})\omega t}}{\sqrt{m!}} m (a^\dagger)^{m-1} |0(t=0)\rangle$$

$$= e^{-i\omega t} \sqrt{m} \frac{e^{-i(m-1+\frac{1}{2})\omega t}}{\sqrt{(m-1)!}}$$

$$(a^\dagger)^{m-1} |0\rangle$$

$$= e^{-i\omega t} \sqrt{m} |m-1\rangle$$

$$e) \quad a|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} a|m\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n} e^{-i\omega t} |n-1\rangle$$

$$= e^{-i\omega t} \alpha e^{-|\alpha|^2/2} \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle$$

$$= e^{-i\omega t} \alpha e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-i\omega t} \alpha |\alpha\rangle$$

$$\Rightarrow \langle \alpha | a | \alpha \rangle = e^{-i\omega t} \alpha \langle \alpha | \alpha \rangle$$

$$\begin{aligned}
 \langle \alpha | \alpha \rangle &= e^{-|\alpha|^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^m \alpha^{*n}}{\sqrt{m!} \sqrt{n!}} \underbrace{\langle m | n \rangle}_{=\delta_{mn}} \\
 &= e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = e^{-|\alpha|^2} e^{|\alpha|^2} = \underline{\underline{1}}.
 \end{aligned}$$

$$\Rightarrow \underline{\underline{\langle \alpha | a | \alpha \rangle = e^{-i\omega t} \alpha}}$$

$$\begin{aligned}
 \underline{\underline{\langle \alpha | a^\dagger | \alpha \rangle}} &= \langle \alpha | a | \alpha \rangle^* \\
 &= (e^{-i\omega t} \alpha)^* = \underline{\underline{e^{i\omega t} \alpha^*}}
 \end{aligned}$$

$$\begin{aligned}
 f) \quad \underline{\underline{\langle \alpha | q | \alpha \rangle}} &= \langle \alpha | \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) | \alpha \rangle \\
 &= \sqrt{\frac{\hbar}{2m\omega}} (e^{i\omega t} \alpha^* + e^{-i\omega t} \alpha) \\
 &= \sqrt{\frac{2\hbar}{m\omega}} \frac{1}{2} (e^{i\omega t - i\theta} + e^{-i\omega t + i\theta}) |\alpha| \\
 &= \sqrt{\frac{2\hbar}{m\omega}} |\alpha| \cos(\omega t - \theta) \\
 &= \underline{\underline{q_0 \cos(\omega t - \theta)}}
 \end{aligned}$$

$$\text{where } \underline{\underline{q_0 = \sqrt{\frac{2\hbar}{m\omega}} |\alpha|}}$$

$$\underline{\underline{\langle \alpha | p | \alpha \rangle}} = \langle \alpha | i \sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a) | \alpha \rangle$$

$$= i \sqrt{\frac{m\hbar\omega}{2}} \langle \alpha | (a^\dagger - a) | \alpha \rangle$$

$$= - \sqrt{\frac{m\hbar\omega}{2}} \frac{1}{i} (e^{i\omega t} \alpha^* - e^{-i\omega t} \alpha)$$

$$= - \sqrt{2m\hbar\omega} \frac{1}{2i} (e^{i\omega t - i\theta} - e^{-i\omega t + i\theta}) |\alpha|$$

$$= - \sqrt{2m\hbar\omega} |\alpha| \sin(\omega t - \theta)$$

$$\underline{\underline{= p_0 \sin(\omega t - \theta)}}$$

where

$$\underline{\underline{p_0 = - \sqrt{2m\hbar\omega} |\alpha|}}$$

g) We need to show that postulating a state with the property

$$a^\dagger |\beta\rangle = |\beta\rangle$$

leads to a contradiction:

$$1. \quad \langle \beta | a a^\dagger | \beta \rangle = \underline{\underline{|\beta|^2 \langle \beta | \beta \rangle}}$$

$$2. \quad \langle \beta | a a^\dagger | \beta \rangle = \langle \beta | a^\dagger a | \beta \rangle + \langle \beta | \beta \rangle$$

$$= \langle \beta | a a^\dagger | \beta \rangle + \langle \beta | \beta \rangle$$

$$= |\beta|^2 \langle \beta | \beta \rangle^* + \langle \beta | \beta \rangle$$

$$= |\beta|^2 \langle \beta | \beta \rangle + \langle \beta | \beta \rangle = (|\beta|^2 + 1) \langle \beta | \beta \rangle$$

Hence we have that

$$\underline{|\beta|^2 \langle \beta | \beta \rangle = (|\beta|^2 + 1) \langle \beta | \beta \rangle}$$

when setting  $1. = 2.$

This is contradictory unless  $|\beta\rangle = 0.$

h)  $a$  is not hermitean.

An operator  $O$  which is hermitean fulfills the relation

$$O^\dagger = O$$

Hence, if  $|\sigma_1\rangle$  and  $|\sigma_2\rangle$  are eigenstates for  $O$ :

$$O|\sigma_1\rangle = \sigma_1|\sigma_1\rangle \quad \text{and} \\ O|\sigma_2\rangle = \sigma_2|\sigma_2\rangle$$



Hence,

$$\langle \sigma_1 | \sigma | \sigma_1 \rangle = \sigma_1 \langle \sigma_1 | \sigma_1 \rangle$$

$$\begin{aligned} \langle \sigma_1 | \sigma | \sigma_1 \rangle &= \langle \sigma_1 | \sigma^\dagger | \sigma_1 \rangle^* \\ &= \langle \sigma_1 | \sigma | \sigma_1 \rangle^* = \sigma_1^* \langle \sigma_1 | \sigma_1 \rangle \end{aligned}$$

$\Rightarrow \underline{\sigma_1 = \sigma_1^*}$  The eigenvalues are real.

Suppose  $\sigma_1 \neq \sigma_2$

$$\langle \sigma_1 | \sigma | \sigma_2 \rangle = \sigma_2 \langle \sigma_1 | \sigma_2 \rangle$$

$$\begin{aligned} \langle \sigma_1 | \sigma | \sigma_2 \rangle &= \langle \sigma_2 | \sigma^\dagger | \sigma_1 \rangle^* \\ &= \langle \sigma_2 | \sigma | \sigma_1 \rangle^* = \sigma_1^* \langle \sigma_2 | \sigma_1 \rangle^* \\ &= \sigma_1 \langle \sigma_1 | \sigma_2 \rangle. \end{aligned}$$

$\Rightarrow$

$$\underline{\sigma_1 \langle \sigma_1 | \sigma_2 \rangle = \sigma_2 \langle \sigma_1 | \sigma_2 \rangle}$$

$$\Rightarrow \underline{\underline{\langle \sigma_1 | \sigma_2 \rangle = 0}}$$



Problem 2

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a) The density operator is defined as

$$\rho = \sum_{\alpha=1}^{N_1} |\alpha\rangle W_{\alpha} \langle\alpha|$$

Where  $|\alpha\rangle$  is the pure state of the sub system. In our case, there are two sub systems:  $j=1$  and  $2$ . Both are in the ground state:  $|1,1\rangle$  and  $|1,2\rangle$ .

We do not know whether we have a bit of size 1 or 2:  $W_1 = W_2 = 1/2$ ; minimum information.

Hence, we have

$$\rho = \frac{1}{2} (|1,1\rangle \langle 1,1| + |1,2\rangle \langle 1,2|)$$

b)

$$\underline{\underline{\rho(x, x') = \langle x | \rho | x' \rangle}}$$

$$= \frac{1}{2} \{ \langle x | 1,1 \rangle \langle 1,1 | x' \rangle + \langle x | 1,2 \rangle \langle 1,2 | x' \rangle \}$$

$$\underline{\underline{= \frac{1}{2} \{ \psi_{1,1}(x) \psi_{1,1}^*(x') + \psi_{1,2}(x) \psi_{1,2}^*(x') \}}}$$

c)

$$\underline{\langle H \rangle} = \text{Trace}(\rho H)$$

$$= \int_{-\infty}^{+\infty} dx \langle x | \rho H | x \rangle$$

$$= \int_{-\infty}^{+\infty} dx \langle x | \frac{1}{2} (|1,1\rangle \langle 1,1| H + |1,2\rangle \langle 1,2| H) | x \rangle$$

$$= \int_{-\infty}^{+\infty} dx \langle x | \frac{1}{2} (|1,1\rangle \langle 1,1| E_1 + |1,2\rangle \langle 1,2| E_2) | x \rangle$$

$$= \int_{-\infty}^{+\infty} dx \frac{1}{2} (E_1 \psi_{1,1}^*(x) \psi_{1,1}(x)$$

$$+ E_2 \psi_{1,2}^*(x) \psi_{1,2}(x))$$

$$= \frac{1}{2} (E_1 + E_2) = \frac{1}{2} \frac{\pi^2 \hbar^2}{2m} \left( \frac{1}{L_1^2} + \frac{1}{L_2^2} \right)$$

$$= \underline{\underline{\frac{\pi^2 \hbar^2}{4m} \left( \frac{1}{L_1^2} + \frac{1}{L_2^2} \right)}}$$