Information

Department of Physics

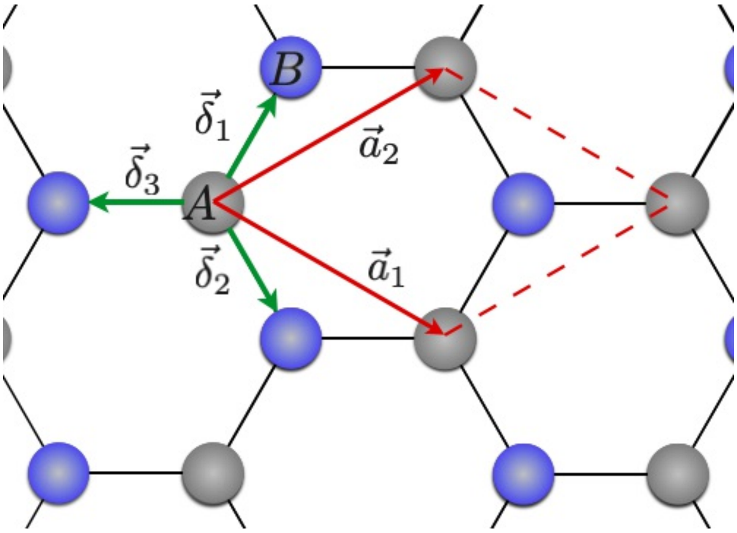
Examination paper for TFY4210/FY8916 Quantum theory of many-particle systems

Academic contact during examination: Øyvind Sande Borck
Phone: 408 59 107
Examination date: 22.05.2019
Examination time (from-to): 15 - 19
Permitted examination support material: Support material code C:
Approved calculator
Rottman: Matematisk formelsamling
Barnett & Cronin: Mathematical Formulae

Other information:

Students will find the examination results in Studentweb. Please contact the department if you have questions about your results. The Examinations Office will not be able to answer this.

1 Problem 1-1



In the lectures we discussed a tight-binding approximation (TBA) for graphene (see figure above) with a Hamiltonian given by

$$H = t\sum_{j}\sum_{l=1}^{3}\left(a_j^\dagger b_{j+l} + b_{j+l}^\dagger a_j\right)$$

Questions:

Give **short** answers to each of the questions below:

a) Which assumptions are made in this approximation?

b) Define/explain the symbols t, $a_j^\dagger$, a_j, b_{j+l} and $b_{j+l}^\dagger$ that goes into this Hamiltonian.

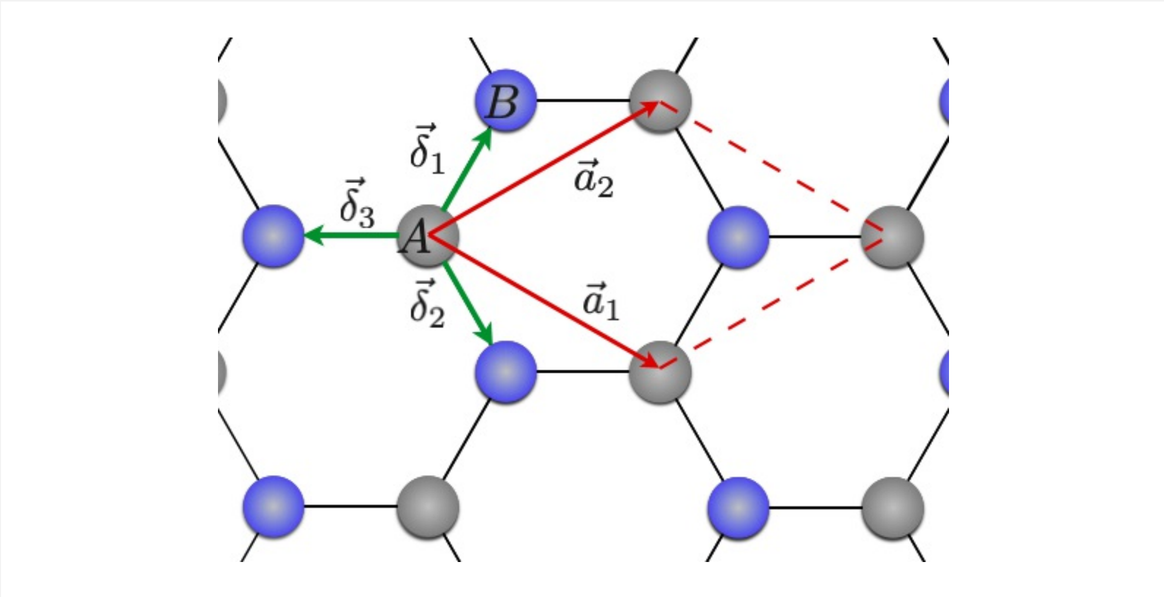
Fill in your answer here

Format

Words: 0

Maximum marks: 4

Problem 1-2



We continue our study of the Hamiltonian:

$$H = t\sum_j \sum_{l=1}^{3} \left(a_j^\dagger b_{j+l} + b_{j+l}^\dagger a_j\right)$$

Assume periodic boundary conditions, introduce new operators $a_\mathbf{k}, b_\mathbf{k}$ via a Fourier transform

$$a_j = \frac{1}{\sqrt{N}} \sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r_j}} a_\mathbf{k}$$

$$b_j = \frac{1}{\sqrt{N}} \sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}_j} b_\mathbf{k}$$

where N is the number of unit cells, and show that the Hamiltonian can be written in the form

$$H = \sum_\mathbf{k} \left(a_\mathbf{k}^\dagger, b_\mathbf{k}^\dagger\right) \begin{pmatrix} 0 & tS(\mathbf{k}) \\ tS^*(\mathbf{k}) & 0 \end{pmatrix} \begin{pmatrix} a_\mathbf{k} \\ b_\mathbf{k} \end{pmatrix}$$

Here $S(\mathbf{k}) = \sum_{l=1}^{3} e^{i\mathbf{k}\cdot\boldsymbol{\delta}_l}$, where $\boldsymbol{\delta}_l$ are nearest neighbour vectors (see figure above).

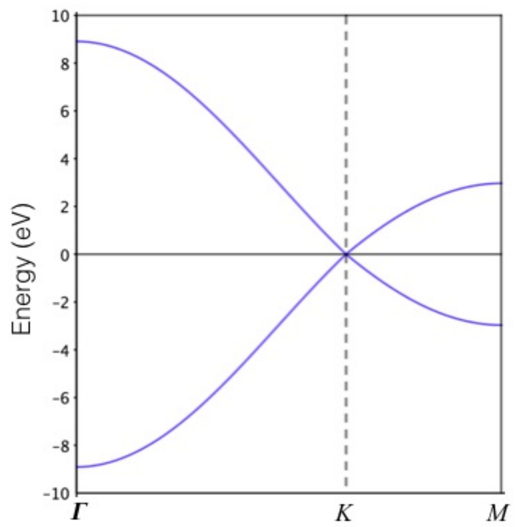
Fill in your answer here

Words: 0

Maximum marks: 10

3

Problem 1-3



The figure above shows a plot of the band structure of graphene calculated within this TBA. We shall consider the electronic states near the K-point. In the 'first-quantized' formalism, the tight-binding Hamiltonian for graphene is

$$h(\mathbf{k}) = \begin{pmatrix} 0 & tS(\mathbf{k}) \\ tS^*(\mathbf{k}) & 0 \end{pmatrix}$$

In the lecures it was shown that close to the *K*-point

$$S(\mathbf{q}) \approx \frac{3a}{2}(q_x - iq_y)$$

where $\mathbf{q}$ is a small wave-vector near $\mathbf{K}$, that is $\mathbf{k} = \mathbf{K} + \mathbf{q}$ and $|\mathbf{q}| \ll |\mathbf{K}|$.

Problem:

Show that the effective Hamiltonian for states near $\mathbf{K}$ takes the form

$$h(\mathbf{q}) = \hbar v_{\mathrm{F}} \left(\sigma_x q_x + \sigma_y q_y\right)$$

where σ_x and σ_y are the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Give the expression for v_{F} .

Fill in your answer here

Format | B I U x_2 x^2 | I_x | | | Ω | | Σ | ABC

Words: 0

Maximum marks: 6

4

Problem 2-1

Consider the Dirac equation for a free spin$-\frac{1}{2}$ particle in three spatial dimensions

$$(i\hbar\gamma^\mu\partial_\mu - mc)\,\Psi = 0$$

where γ^μ are the four Dirac $\gamma-$matrices, $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$ is the covariant four-derivative and Ψ is a four-component spinor.

Problem:

Express Ψ in terms of two two-compent spinors φ_A and φ_B, and show that for a massless particle ($m = 0$), the Dirac equation can be decoupled into the two *Weyl equations*

$$i\hbar\frac{\partial\varphi_\pm}{\partial t} = \pm c\boldsymbol{\sigma}\cdot\mathbf{p}\varphi_\pm$$

where $\varphi_\pm \equiv \varphi_A \pm \varphi_B$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices, and $\mathbf{p} = -i\hbar\nabla$.

Fill in your answer here

Format B I U x_2 x^2 I_x Ω Σ

Words: 0

Maximum marks: 10

5

Problem 2-2

In two spatial dimension the Weyl-equation for φ_+ can be expressed in terms of only two of the Pauli matrices, for example:

$$i\hbar\frac{\partial\varphi_+}{\partial t} = c(\sigma_x p_x + \sigma_y p_y)\varphi_+$$

Problems:

a) Show that the energy eigenvalues for this equation are $E_\pm = \pm c|\mathbf{p}|$

b) Compare the Hamiltonian in this equation with the effective hamiltonian for graphene near the K-point in problem 1-3 and comment.

Fill in your answer here

Format B I U x_2 x^2 I_x Ω Σ

Words: 0

Maximum marks: 10

Problem 2-2 (Correction)

In two spatial dimension the Weyl-equation for φ_+ can be expressed in terms of only two of the Pauli matrices, for example:

$$i\hbar\frac{\partial\varphi_+}{\partial t} = c(\sigma_x p_x + \sigma_y p_y)\varphi_+$$

Problems:

a) Show that the energy eigenvalues for this equation are $E_\pm = \pm c|\mathbf{p}|$

for a free particle with plane-wave solutions of the form

$$\varphi_+ = Ne^{-\frac{i}{\hbar}(Et-\mathbf{p}\cdot\mathbf{r})}$$

6

Problem 3-1

Consider a system of noninteracting electrons. In the 'second-quantized' formalism the Hamiltonian is given by

$$H_0 = \sum_\nu \xi_\nu c_\nu^\dagger c_\nu$$

where ν is a set of quantum numbers, $\xi_\nu = \varepsilon_\nu - \mu$, and μ is the chemical potential.
The Matsubara single-particle Green function is defined as

$$\mathcal{G}(\nu,\tau) = -\langle T_\tau(c_\nu(\tau)c_\nu^\dagger(0))\rangle$$

Here $\tau \in \langle -\beta, \beta\rangle$, (where $\beta = 1/(k_\mathrm{B}T)$ is the inverse temperature), and T_τ is a time-ordering operator which orders operators with earliest (imaginary) time τ to the right, introducing a minus sign when reordering is needed. The time-dependence of the operators is given by

$$c_\nu(\tau) = e^{-H_o\tau}c_\nu e^{H_o\tau} = e^{-\xi_\nu\tau}c_\nu$$

The Fourier transform of $\mathcal{G}(\nu,\tau)$ is given by

$$\mathcal{G}(\nu, ip_n) = \int_0^\beta d\tau\, e^{ip_n\tau}\mathcal{G}(\nu,\tau)$$

where $p_n = (2n+1)\pi/\beta$ is a fermionic Matsubara frequency ($n \in \mathbb{Z}$).

Problem:

Calculate the Matsubara Green function for noninteracting electrons, $\mathcal{G}^{(0)}(\nu, ip_n)$, and use the result to obtain the retarded Green function $G_0^R(\nu,\omega)$.

Fill in your answer here

Format B I U x_2 x^2 I_x Ω Σ

Words: 0

Maximum marks: 10

7

Problem 3-2

Consider electrons scattering with (randomly distributed) impurities in a metal. Neglect the electron spin. In the lecures we developed a perturbative expansion for the single-particle Matsubara Green function $\mathcal{G}(\mathbf{k},\mathbf{k}';ip_m)$ for such a system. By averaging over the position of the impurities, we obtained a Green function which was diagonal in $\mathbf{k}$: $\overline{\mathcal{G}}(\mathbf{k},\mathbf{k}';ip_m) = \overline{\mathcal{G}}(\mathbf{k})\delta_{\mathbf{k}\mathbf{k}'}$.

Question:

Explain why we should expect, on physical grounds, that such an averaging will make the Green function diagonal in $\mathbf{k}$.

Fill in your answer here

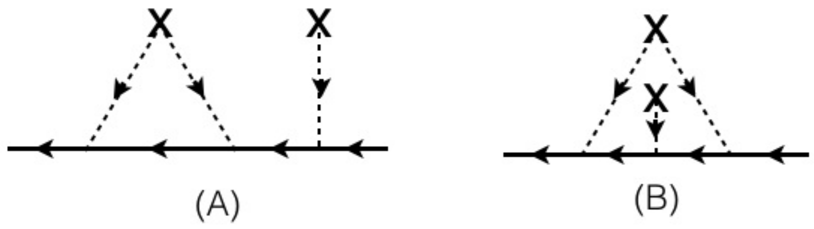
Format

Words: 0

Maximum marks: 4

8

Problem 3-3



The figure above shows two of the Feynman diagrams in the perturbation expansion for $\overline{\mathcal{G}}(\mathbf{k}, ip_m)$.

Questions:

a) For each diagram, determine whether the diagram is reducible or irreducible.
   Justify your answer.

b) Give the mathematical expression for diagram A
   (Do not evalute the wave vector sum).

Fill in your answer here

Format | B I U x_2 x^2 | I_x | Ω | ∑ | ABC

Words: 0

Maximum marks: 6

9

Problem 3-4

The spectral function $A(\mathbf{k},\omega)$ is defined as

$$A(\mathbf{k},\omega) = -\frac{1}{\pi}\mathrm{Im}G^R(\mathbf{k},\omega)$$

where $G^R(\mathbf{k},\omega)$ is the retarded Green function. In the lectures it was shown that in the limit of low impurity density n_{imp} and weak scattering potential u, the retarded Green function is given by

$$\overline{G^R}(\mathbf{k},t) = -i\theta(t)e^{-i(\xi_{\mathbf{k}}+n_{\mathrm{imp}}u)t}e^{-t/(2\tau)}$$

where $1/\tau = 2\pi n_{\mathrm{imp}}u^2 D(0)$ (here D(0) is the density of states at the Fermi level).

Problems

a) Calculate the spectral function for this Green function, and show that it is given by

$$\bar{A}(\mathbf{k},\omega) = \frac{1}{\pi}\frac{1/(2/\tau)}{(\omega-(\xi_{\mathbf{k}}+n_{\mathrm{imp}}u))^2+(1/(2\tau))^2}$$

b) Show that this spectral function satisfy the sum rule:

$$\int_{-\infty}^{\infty} d\omega \bar{A}(\mathbf{k},\omega) = 1$$

Fill in your answer here

Words: 0

Maximum marks: 10