

TFY4210 Quantum theory of many-particle systems
Exam 2015
Solution

Problem 1

$$\begin{aligned} (a) \quad b_{\uparrow} b_{\uparrow}^{\dagger} |n_{\uparrow}, n_{\downarrow}\rangle &= \sqrt{n_{\uparrow}+1} b_{\uparrow} |n_{\uparrow}+1, n_{\downarrow}\rangle \\ &= \sqrt{n_{\uparrow}+1} \sqrt{n_{\uparrow}+1} |(n_{\uparrow}+1)-1, n_{\downarrow}\rangle \\ &= (n_{\uparrow}+1) |n_{\uparrow}, n_{\downarrow}\rangle, \end{aligned}$$

$$\begin{aligned} b_{\uparrow}^{\dagger} b_{\uparrow} |n_{\uparrow}, n_{\downarrow}\rangle &= \sqrt{n_{\uparrow}} b_{\uparrow}^{\dagger} |n_{\uparrow}-1, n_{\downarrow}\rangle \\ &= \sqrt{n_{\uparrow}} \sqrt{(n_{\uparrow}-1)+1} |(n_{\uparrow}-1)+1, n_{\downarrow}\rangle \\ &= n_{\uparrow} |n_{\uparrow}, n_{\downarrow}\rangle \end{aligned}$$

$$\Rightarrow (b_{\uparrow} b_{\uparrow}^{\dagger} - b_{\uparrow}^{\dagger} b_{\uparrow}) |n_{\uparrow}, n_{\downarrow}\rangle$$

$$= (n_{\uparrow}+1 - n_{\uparrow}) |n_{\uparrow}, n_{\downarrow}\rangle = |n_{\uparrow}, n_{\downarrow}\rangle$$

$$\text{i.e. } [b_{\uparrow}, b_{\uparrow}^{\dagger}] |n_{\uparrow}, n_{\downarrow}\rangle = 1 \cdot |n_{\uparrow}, n_{\downarrow}\rangle.$$

As this result is valid for an arbitrary basis state $|n_{\uparrow}, n_{\downarrow}\rangle$, we obtain the operator identity

$$[b_{\uparrow}, b_{\uparrow}^{\dagger}] = 1 \quad \text{QED.}$$

More generally, the commutation relations that follow from (1) can be written ($\alpha, \beta = \uparrow, \downarrow$)

$$[b_{\alpha}, b_{\beta}^{\dagger}] = \delta_{\alpha\beta}, \quad [b_{\uparrow}, b_{\downarrow}] = [b_{\uparrow}^{\dagger}, b_{\downarrow}^{\dagger}] = 0$$

$$\begin{aligned}
 (b) \quad \underline{[\hat{S}^+, \hat{S}^-]} &= b_{\uparrow}^{\dagger} b_{\downarrow} b_{\downarrow}^{\dagger} b_{\uparrow} - b_{\downarrow}^{\dagger} b_{\uparrow} b_{\uparrow}^{\dagger} b_{\downarrow} \\
 &= b_{\uparrow}^{\dagger} (1 + \hat{n}_{\downarrow}) b_{\uparrow} - b_{\downarrow}^{\dagger} (1 + \hat{n}_{\uparrow}) b_{\downarrow} \\
 &= \hat{n}_{\uparrow} (1 + \hat{n}_{\downarrow}) - \hat{n}_{\downarrow} (1 + \hat{n}_{\uparrow}) \\
 &= \hat{n}_{\uparrow} + \cancel{\hat{n}_{\uparrow} \hat{n}_{\downarrow}} - \hat{n}_{\downarrow} - \cancel{\hat{n}_{\downarrow} \hat{n}_{\uparrow}} \\
 &= \hat{n}_{\uparrow} - \hat{n}_{\downarrow} = \underline{2\hat{S}^z},
 \end{aligned}$$

(using bosonic comm. relations including the $\uparrow$ - and $\downarrow$ -operators commute)

$$\begin{aligned}
 \underline{[\hat{S}^z, \hat{S}^+]} &= \left[\frac{1}{2} (\hat{n}_{\uparrow} - \hat{n}_{\downarrow}), b_{\uparrow}^{\dagger} b_{\downarrow} \right] \\
 &= \frac{1}{2} [\hat{n}_{\uparrow}, b_{\uparrow}^{\dagger} b_{\downarrow}] - \frac{1}{2} [\hat{n}_{\downarrow}, b_{\uparrow}^{\dagger} b_{\downarrow}] \\
 &= \frac{1}{2} \underbrace{[\hat{n}_{\uparrow}, b_{\uparrow}^{\dagger}]}_{= b_{\uparrow}^{\dagger}} b_{\downarrow} - \frac{1}{2} b_{\uparrow}^{\dagger} \underbrace{[\hat{n}_{\downarrow}, b_{\downarrow}]}_{= -b_{\downarrow}}
 \end{aligned}$$

using Eqs. (23), (24)

$$= b_{\uparrow}^{\dagger} b_{\downarrow} = \underline{\underline{\hat{S}^+}}$$

$[\hat{S}^z, \hat{S}^-] = -\hat{S}^-$ can be proven in the same way. Alternatively, one can note that since the Schwinger boson expressions for $\hat{S}^+$ and $\hat{S}^-$ are h.c.'s of each other,

$$\begin{aligned}
 \underline{[\hat{S}^z, \hat{S}^-]} &= \hat{S}^z \hat{S}^- - \hat{S}^- \hat{S}^z = (\hat{S}^+ \hat{S}^z)^{\dagger} - (\hat{S}^z \hat{S}^-)^{\dagger} \\
 &= [\hat{S}^+, \hat{S}^z]^{\dagger} = (-\hat{S}^+)^{\dagger} = \underline{\underline{-\hat{S}^-}}
 \end{aligned}$$

$$(c) \quad \hat{\vec{S}}^2 = \hat{S}^x \hat{S}^x + \hat{S}^y \hat{S}^y + \hat{S}^z \hat{S}^z$$

$$= \left(\frac{1}{2}\right)^2 (\hat{S}^+ + \hat{S}^-)(\hat{S}^+ + \hat{S}^-) + \left(\frac{1}{2i}\right)^2 (\hat{S}^+ - \hat{S}^-)(\hat{S}^+ - \hat{S}^-) + \hat{S}^z \hat{S}^z$$

$$= \frac{1}{2} (\hat{S}^+ \hat{S}^- + \hat{S}^- \hat{S}^+) + \hat{S}^z \hat{S}^z$$

$$= \frac{1}{2} (b_{\uparrow}^{\dagger} b_{\downarrow} b_{\downarrow}^{\dagger} b_{\uparrow} + b_{\downarrow}^{\dagger} b_{\uparrow} b_{\uparrow}^{\dagger} b_{\downarrow}) + \left(\frac{1}{2}\right)^2 (\hat{n}_{\uparrow} - \hat{n}_{\downarrow})^2$$

$$= \frac{1}{2} [\hat{n}_{\uparrow} (1 + \hat{n}_{\downarrow}) + \hat{n}_{\downarrow} (1 + \hat{n}_{\uparrow})] + \frac{1}{4} (\hat{n}_{\uparrow} - \hat{n}_{\downarrow})^2$$

$$= \frac{1}{2} [\hat{n}_{\uparrow} + \hat{n}_{\downarrow} + 2\hat{n}_{\uparrow}\hat{n}_{\downarrow}] + \frac{1}{4} [\hat{n}_{\uparrow}^2 - 2\hat{n}_{\uparrow}\hat{n}_{\downarrow} + \hat{n}_{\downarrow}^2]$$

$$= \frac{1}{2} (\hat{n}_{\uparrow} + \hat{n}_{\downarrow}) + \frac{1}{4} (\hat{n}_{\uparrow} + \hat{n}_{\downarrow})^2$$

$$= \frac{\hat{n}_{\uparrow} + \hat{n}_{\downarrow}}{2} \left[\frac{\hat{n}_{\uparrow} + \hat{n}_{\downarrow}}{2} + 1 \right]$$

$$\Rightarrow \hat{\vec{S}}^2 |n_{\uparrow}, n_{\downarrow}\rangle$$

$$= \frac{\hat{n}_{\uparrow} + \hat{n}_{\downarrow}}{2} \left[\frac{\hat{n}_{\uparrow} + \hat{n}_{\downarrow}}{2} + 1 \right] |n_{\uparrow}, n_{\downarrow}\rangle$$

$$= \frac{n_{\uparrow} + n_{\downarrow}}{2} \left[\frac{n_{\uparrow} + n_{\downarrow}}{2} + 1 \right] |n_{\uparrow}, n_{\downarrow}\rangle$$

$$= \underline{\underline{S(S+1) |n_{\uparrow}, n_{\downarrow}\rangle}} \quad \text{if } n_{\uparrow} + n_{\downarrow} = 2S$$

$$(d) \quad U(\theta) = e^{-i \hat{S}^z \theta / \hbar}$$

Setting $\hbar=1$ as usual and using the Schwinger boson representation gives

$$U(\theta) = \exp \left[-i \frac{1}{2} (\hat{n}_\uparrow - \hat{n}_\downarrow) \theta \right]$$

$$\Rightarrow U(\theta) b_\alpha^\dagger U^\dagger(\theta) = e^{-i \frac{\theta}{2} (\hat{n}_\uparrow - \hat{n}_\downarrow)} b_\alpha^\dagger e^{i \frac{\theta}{2} (\hat{n}_\uparrow - \hat{n}_\downarrow)}$$

To calculate this we use the Baker-Hausdorff theorem (25).

$$e^{-B} A e^B = A + [A, B] + \frac{1}{2!} [[A, B], B] + \dots$$

$$\text{with } A = b_\alpha^\dagger \text{ and } B = i \frac{\theta}{2} (\hat{n}_\uparrow - \hat{n}_\downarrow)$$

$$\begin{aligned} \text{For } \alpha = \uparrow, \quad [A, B] &= [b_\uparrow^\dagger, i \frac{\theta}{2} (\hat{n}_\uparrow - \hat{n}_\downarrow)] \\ &= i \frac{\theta}{2} [b_\uparrow^\dagger, \hat{n}_\uparrow] = -i \frac{\theta}{2} b_\uparrow^\dagger, \end{aligned}$$

where we used (24). As the result is just a constant $-i \frac{\theta}{2}$ times A , it follows that

$$[A, B]_n = \left(-i \frac{\theta}{2} \right)^n b_\uparrow^\dagger$$

Therefore

$$\underline{U(\theta) b_\uparrow^\dagger U^\dagger(\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i \frac{\theta}{2} \right)^n b_\uparrow^\dagger = \underline{e^{-i \frac{\theta}{2}} b_\uparrow^\dagger}$$

$$\text{For } \alpha = \downarrow, \quad [A, B] = i \frac{\theta}{2} [b_\downarrow^\dagger, -\hat{n}_\downarrow] = i \frac{\theta}{2} b_\downarrow^\dagger,$$

$$\text{thus giving } \underline{U(\theta) b_\downarrow^\dagger U^\dagger(\theta)} = \underline{e^{i \frac{\theta}{2}} b_\downarrow^\dagger}$$

Use that $|S, m\rangle = |n_{\uparrow}, n_{\downarrow}\rangle$ with $n_{\uparrow} = S \pm m$

and $|n_{\uparrow}, n_{\downarrow}\rangle \propto (b_{\uparrow}^{\dagger})^{n_{\uparrow}} (b_{\downarrow}^{\dagger})^{n_{\downarrow}} |0,0\rangle$

$$\Rightarrow U(2\pi) |S, m\rangle \propto U(2\pi) (b_{\uparrow}^{\dagger})^{S+m} (b_{\downarrow}^{\dagger})^{S-m} |0,0\rangle$$

$$= U(2\pi) \underbrace{b_{\uparrow}^{\dagger} b_{\uparrow}^{\dagger} \dots b_{\uparrow}^{\dagger}}_{S+m \text{ factors}} \underbrace{b_{\downarrow}^{\dagger} b_{\downarrow}^{\dagger} \dots b_{\downarrow}^{\dagger}}_{S-m \text{ factors}} |0,0\rangle$$

Now insert $I = U^{\dagger}(2\pi) U(2\pi)$ between adjacent $b_{\alpha}^{\dagger}$ operators and after the last $b_{\downarrow}^{\dagger}$

$$\Rightarrow \left(U(2\pi) b_{\uparrow}^{\dagger} U^{\dagger}(2\pi) \right)^{S+m} \left(U(2\pi) b_{\downarrow}^{\dagger} U^{\dagger}(2\pi) \right)^{S-m} U(2\pi) |0,0\rangle$$

$$\text{Use } U(2\pi) b_{\uparrow}^{\dagger} U^{\dagger}(2\pi) = e^{-i\frac{2\pi}{2}} b_{\uparrow}^{\dagger} = e^{-i\pi} b_{\uparrow}^{\dagger} = -b_{\uparrow}^{\dagger}$$

$$U(2\pi) b_{\downarrow}^{\dagger} U^{\dagger}(2\pi) = e^{i\frac{2\pi}{2}} b_{\downarrow}^{\dagger} = e^{i\pi} b_{\downarrow}^{\dagger} = -b_{\downarrow}^{\dagger}$$

$$U(2\pi) |0,0\rangle = e^{-i\frac{2\pi}{2}(\hat{n}_{\uparrow} - \hat{n}_{\downarrow})} = \sum_{m=0}^{\infty} \frac{(-i\pi)^m}{m!} (\hat{n}_{\uparrow} - \hat{n}_{\downarrow})^m |0,0\rangle$$

$$= |0,0\rangle \quad \text{since only the } m=0 \text{ term gives a non-zero contribution (since } \hat{n}_{\uparrow} |0,0\rangle = \hat{n}_{\downarrow} |0,0\rangle = 0)$$

$$\Rightarrow U(2\pi) |S, m\rangle \propto (-b_{\uparrow}^{\dagger})^{S+m} (-b_{\downarrow}^{\dagger})^{S-m} |0,0\rangle$$

$$= (-1)^{S+m+S-m} (b_{\uparrow}^{\dagger})^{S+m} (b_{\downarrow}^{\dagger})^{S-m} |0,0\rangle$$

$$= (-1)^{2S} (b_{\uparrow}^{\dagger})^{S+m} (b_{\downarrow}^{\dagger})^{S-m} |0,0\rangle$$

$$\text{i.e. } U(2\pi) |S, m\rangle = (-1)^{2S} |S, m\rangle$$

which is what we wanted to verify using the Schvinger boson representation,

Remarks:

- Depending on at what stage one inserts the value $\theta = 2\pi$, one may end up with the factor $(-1)^{2m}$ instead of $(-1)^{2S}$. But these have the same value, since if $2S$ is an even (odd) integer, then $2m$ is also always even (odd).
- The result $U(2\pi) |S, m\rangle = (-1)^{2S} |S, m\rangle$ is trivial to verify from the general relations:

$$U(2\pi) |S, m\rangle = \exp(-i \cdot 2\pi \hat{S}_z) |S, m\rangle$$

$$= \exp(-i \cdot 2\pi m) |S, m\rangle = (-1)^{2m} |S, m\rangle = (-1)^{2S} |S, m\rangle$$

Problem 2

$$\begin{aligned} (a) \quad c_{k+2\pi m} &= \frac{1}{\sqrt{N}} \sum_j e^{-i(k+2\pi m)j} c_j \\ &= \frac{1}{\sqrt{N}} \sum_j e^{-ikj} c_j \underbrace{e^{-i2\pi mj}}_{=1 \text{ since both } m \text{ and } j} = c_k \quad (*) \end{aligned}$$

Periodic BC's gives $k = 2\pi m/N$ (m integer). Pick k 's to lie in region of length 2π (cf. (*)); we choose the standard region $-\pi \leq k < \pi$ (1B2)

Let us define $H \equiv H_t + H_\Delta$

$$\begin{aligned} H_t &= -t \sum_j (c_j^\dagger c_{j+1} + \text{h.c.}) \\ &= -t \sum_j \left(\frac{1}{\sqrt{N}}\right)^2 \sum_{k,k'} e^{-ik'j} e^{ik(j+1)} c_{k'}^\dagger c_k + \text{h.c.} \\ &= -t \sum_{k,k'} c_{k'}^\dagger c_k e^{ik} \underbrace{\frac{1}{N} \sum_j e^{i(k-k')j}}_{=\delta_{kk'}} + \text{h.c.} \\ &\quad \leftarrow \text{from using (28) with } F(k-k') = k-k' \\ &= -t \sum_k (e^{ik} + e^{-ik}) c_k^\dagger c_k = \underline{\underline{\sum_k \varepsilon_k c_k^\dagger c_k}} \end{aligned}$$

where $\varepsilon_k = -2t \cos k$

$$\begin{aligned} \text{We have } \varepsilon_{k\pm\pi} &= -2t \cos(k\pm\pi) \\ &= -2t \left[\cos k \underbrace{\cos \pi}_{=-1} \mp \sin k \underbrace{\sin \pi}_0 \right] = -\varepsilon_k \end{aligned}$$

$$\begin{aligned}
H_{\Delta} &= \Delta \sum_j (-1)^j c_j^{\dagger} c_j \quad (\text{use } -1 = e^{i\pi}) \\
&= \Delta \sum_j e^{i\pi j} \left(\frac{1}{\sqrt{N}}\right)^2 \sum_{kk'} e^{-ik'j} e^{ikj} c_{k'}^{\dagger} c_k \\
&= \Delta \sum_{kk'} c_{k'}^{\dagger} c_k \frac{1}{N} \sum_j e^{ij(k-k'+\pi)}
\end{aligned}$$

To calculate the j -sum we use (28) with $F(k-k') = k - k' + \pi$. The sum is thus 0 unless $k - k' + \pi = 2\pi m$ (m integer)

$$\Rightarrow k - k' = (2m-1)\pi$$

Given that both k and k' are in $1\mathbb{B}\mathbb{Z}$, solutions are only possible for $m=0,1$, i.e. for $k - k' = \pm\pi$. Thus we may write

$$H_{\Delta} = \Delta \sum_{kk'} c_{k'}^{\dagger} c_k \sum_{p=\pm 1} \delta_{k', k+p\pi}$$

For terms with $k \in [-\pi, 0)$ we only get a contribution from $k' = k + \pi$ ($p = +1$)

$$\Rightarrow \Delta c_{k+\pi}^{\dagger} c_k$$

For terms with $k \in [0, \pi)$ we only get a contribution from $k' = k - \pi$ ($p = -1$)

$$\Rightarrow \Delta c_{k-\pi}^{\dagger} c_k = \Delta c_{k+\pi}^{\dagger} c_k$$

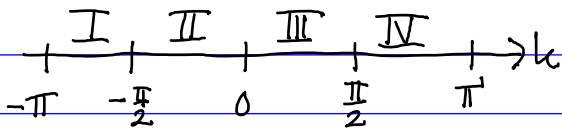
$\uparrow$ using $c_{k+\pi} = c_{k-\pi}$ which follows from $c_{k+2\pi} = c_k$

$$\Rightarrow H_{\Delta} = \Delta \sum_k c_{k+\pi}^{\dagger} c_k$$

Thus we have shown that

$$\underline{\underline{H = \sum_{k \in 1\mathbb{B}\mathbb{Z}} \left[\epsilon_k c_k^{\dagger} c_k + \Delta c_{k+\pi}^{\dagger} c_k \right]}}$$

(b) It is convenient to divide the 1BZ into 4 regions I, II, III, IV as shown below:



$$\Rightarrow H = \left(\sum_{k \in \text{I}} + \sum_{k \in \text{II}} + \sum_{k \in \text{III}} + \sum_{k \in \text{IV}} \right) \left(\varepsilon_k c_k^\dagger c_k + \Delta c_{k+\pi}^\dagger c_k \right)$$

Regions II+III make up the MBZ, and we see that their contribution gives the first and fourth terms in (16). The contribution from region I is

$$\begin{aligned} & \sum_{k \in \text{I}} \left(\varepsilon_k c_k^\dagger c_k + \Delta c_{k+\pi}^\dagger c_k \right) \\ &= \sum_{k \in \text{III}} \left[\underbrace{\varepsilon_{k-\pi}}_{-\varepsilon_k} \underbrace{c_{k-\pi}^\dagger}_{c_{k+\pi}^\dagger} \underbrace{c_{k-\pi}}_{c_{k+\pi}} + \Delta c_{k+\pi-\pi}^\dagger \underbrace{c_{k-\pi}}_{c_{k+\pi}} \right] \\ &= \sum_{k \in \text{III}} \left[-\varepsilon_k c_{k+\pi}^\dagger c_{k+\pi} + \Delta c_k^\dagger c_{k+\pi} \right] \end{aligned}$$

Similarly, the contribution from region IV is

$$\begin{aligned} & \sum_{k \in \text{IV}} \left(\varepsilon_k c_k^\dagger c_k + \Delta c_{k+\pi}^\dagger c_k \right) \\ &= \sum_{k \in \text{II}} \left(\underbrace{\varepsilon_{k+\pi}}_{-\varepsilon_k} c_{k+\pi}^\dagger c_{k+\pi} + \Delta \underbrace{c_{k+\pi+\pi}^\dagger}_{c_k^\dagger} c_{k+\pi} \right) \\ &= \sum_{k \in \text{II}} \left[-\varepsilon_k c_{k+\pi}^\dagger c_{k+\pi} + \Delta c_k^\dagger c_{k+\pi} \right] \end{aligned}$$

Thus the contributions from regions I+IV give the second and third terms in (16). In conclusion,

$$H = \sum_{k \in \text{MBZ}} \left[\varepsilon_k (c_k^\dagger c_k - c_{k+\pi}^\dagger c_{k+\pi}) + \Delta (c_k^\dagger c_{k+\pi} + c_{k+\pi}^\dagger c_k) \right]$$

(c) Each term in the k -sum can now be diagonalized separately. To save some writing we drop the k -index in the following and also define $c_k \equiv a$, $c_{k+\pi} \equiv b$. Thus we will diagonalize the operator

$$h \equiv \varepsilon (a^\dagger a - b^\dagger b) + \Delta (a^\dagger b + b^\dagger a)$$

$$\begin{aligned} a^\dagger a &= (u\alpha^\dagger - v\beta^\dagger)(u\alpha - v\beta) \\ &= u^2 \alpha^\dagger \alpha + v^2 \beta^\dagger \beta - uv(\alpha^\dagger \beta + \beta^\dagger \alpha) \end{aligned}$$

$$\begin{aligned} b^\dagger b &= (v\alpha^\dagger + u\beta^\dagger)(v\alpha + u\beta) \\ &= v^2 \alpha^\dagger \alpha + u^2 \beta^\dagger \beta + uv(\alpha^\dagger \beta + \beta^\dagger \alpha) \end{aligned}$$

$$\begin{aligned} a^\dagger b &= (u\alpha^\dagger - v\beta^\dagger)(v\alpha + u\beta) \\ &= uv(\alpha^\dagger \alpha - \beta^\dagger \beta) + u^2 \alpha^\dagger \beta - v^2 \beta^\dagger \alpha \end{aligned}$$

$$\begin{aligned} b^\dagger a &= (a^\dagger b)^\dagger \\ &= uv(\alpha^\dagger \alpha - \beta^\dagger \beta) - v^2 \alpha^\dagger \beta + u^2 \beta^\dagger \alpha \end{aligned}$$

$$\Rightarrow h = F(\alpha^\dagger \alpha - \beta^\dagger \beta) + G(\alpha^\dagger \beta + \beta^\dagger \alpha)$$

$$\begin{aligned} \text{where } F &= \varepsilon(u^2 - v^2) + 2\Delta uv \\ G &= -2\varepsilon uv + \Delta(u^2 - v^2) \end{aligned}$$

The off-diagonal terms (i.e. the terms prop. to $(\alpha^\dagger \beta + \beta^\dagger \alpha)$) vanish if $G = 0 \Rightarrow \Delta(u^2 - v^2) = \varepsilon \cdot 2uv$

Using the parametrization $u = \cos \theta$, $v = \sin \theta$, and (29)-(30), the condition $G = 0$ becomes $\Delta \cos 2\theta = \varepsilon \sin 2\theta$

$$\Rightarrow \tan 2\theta = \frac{\Delta}{\varepsilon}$$

This gives furthermore

$$F = \varepsilon \cos 2\theta + \Delta \sin 2\theta = \cos 2\theta [\varepsilon + \Delta \tan 2\theta]$$

$$\text{Using (31) gives } \cos 2\theta = \pm \frac{1}{\sqrt{1 + \tan^2 2\theta}} = \pm \frac{1}{\sqrt{1 + (\Delta/\varepsilon)^2}}$$

Let us choose the positive sign (choosing the negative sign would however also be allowed and would not affect the physical results).

$$\Rightarrow F = \frac{1}{\sqrt{1 + \left(\frac{\Delta}{\varepsilon}\right)^2}} \left[\varepsilon + \Delta \frac{\Delta}{\varepsilon} \right] = \frac{\varepsilon^2 + \Delta^2}{\varepsilon \sqrt{1 + \left(\frac{\Delta}{\varepsilon}\right)^2}}$$

$$= -\sqrt{\varepsilon^2 + \Delta^2} \quad (\text{where we used that } \varepsilon = -|\varepsilon| \text{ for } k \in \text{MBZ})$$

This gives

$$H = \sum_{k \in \text{MBZ}} \left[E_k^{(\alpha)} \alpha_k^\dagger \alpha_k + E_k^{(\beta)} \beta_k^\dagger \beta_k \right]$$

where $E_k^{(\beta)} = -E_k^{(\alpha)} = \sqrt{\varepsilon_k^2 + \Delta^2}$

Here $E_k^{(\alpha)}$ ($E_k^{(\beta)}$) is the energy of the α -mode $\alpha_k^\dagger |0\rangle$ (β -mode $\beta_k^\dagger |0\rangle$) with wave vector k (here $|0\rangle$ is the vacuum state with no fermions).

(d) As there are N wavevectors in 1BZ , there are $N/2$ wavevectors in MBZ . Thus there are $N/2$ α -modes and $N/2$ β -modes. Furthermore, for the generic case $t > 0$ and $\Delta \neq 0$, $\sqrt{\varepsilon_k^2 + \Delta^2} > 0$ for all $k \in \text{MBZ}$. Thus the energy of any β -mode is higher than that of any α -mode. Invoking also the Pauli principle, it follows that in a system with $N/2$ fermions, the ground state $|G\rangle$ is characterized by each α -mode being occupied by one fermion while all β -modes are unoccupied. The ground state energy is therefore

$$E_G = \sum_{k \in \text{MBZ}} E_k^{(\alpha)} = - \sum_{k \in \text{MBZ}} \sqrt{\varepsilon_k^2 + \Delta^2}$$

(The ground state $|G\rangle$ can be written $|G\rangle = \left(\prod_{k \in \text{MBZ}} \alpha_k^\dagger \right) |0\rangle$,

$$\begin{aligned}
 (e) \quad \hat{N}_{\text{even}} - \hat{N}_{\text{odd}} &= \sum_j (-1)^j c_j^\dagger c_j = \frac{1}{\Delta} H_\Delta \\
 &= \sum_{k \in \text{MBZ}} \left[c_k^\dagger c_{k+\pi} + c_{k+\pi}^\dagger c_k \right] \\
 &= \sum_{k \in \text{MBZ}} \left[2u_k v_k (\alpha_k^\dagger \alpha_k - \beta_k^\dagger \beta_k) + (u_k^2 - v_k^2) (\alpha_k^\dagger \beta_k + \text{h.c.}) \right]
 \end{aligned}$$

Given the form of the ground state as discussed in (d), the ground state expectation values are

$$\langle \alpha_k^\dagger \alpha_k \rangle = 1, \quad \langle \beta_k^\dagger \beta_k \rangle = \langle \alpha_k^\dagger \beta_k \rangle = \langle \beta_k^\dagger \alpha_k \rangle = 0$$

$$\Rightarrow \langle (\hat{N}_{\text{even}} - \hat{N}_{\text{odd}}) \rangle = \sum_{k \in \text{MBZ}} 2u_k v_k = \sum_{k \in \text{MBZ}} \sin 2\theta_k$$

$$= \sum_{k \in \text{MBZ}} \cos 2\theta_k \tan 2\theta_k = \sum_{k \in \text{MBZ}} \frac{1}{\sqrt{1 + (\Delta/\epsilon_k)^2}} \frac{\Delta}{\epsilon_k}$$

$$= - \sum_{k \in \text{MBZ}} \frac{\Delta}{\sqrt{\epsilon_k^2 + \Delta^2}}$$

(where we again used that $\epsilon_k = -|\epsilon_k|$ for $k \in \text{MBZ}$)

$$(i) \quad |\Delta| \ll t \Rightarrow |\langle (\hat{N}_{\text{even}} - \hat{N}_{\text{odd}}) \rangle| \text{ small } (\propto \frac{|\Delta|}{t})$$

This is reasonable as in this case there is no appreciable energy difference between even and odd sites.

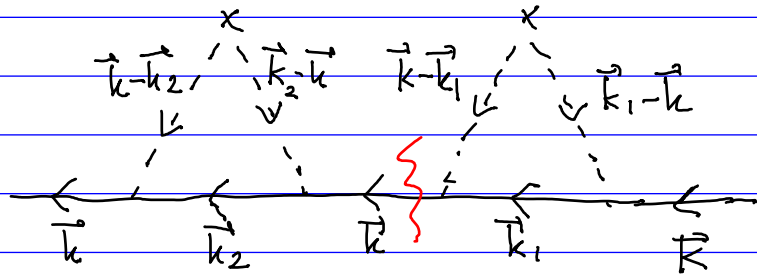
$$|\Delta| \gg t \Rightarrow |\langle (\hat{N}_{\text{even}} - \hat{N}_{\text{odd}}) \rangle| \approx \frac{N}{2}$$

Reasonable; now strong preference for those $N/2$ sites with smallest energy cost, leaving the other $\frac{N}{2}$ sites empty.

(ii) The sign equals $-\text{sign}(\Delta)$: reasonable, as it means odd sites preferred for $\Delta > 0$, even sites for $\Delta < 0$.

Problem 3

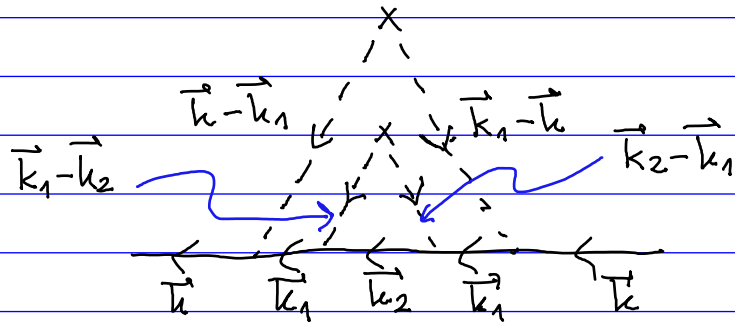
(a) Left diagram:



$$= N^2 [g^{(0)}(\vec{k})]^3 \sum_{\vec{k}_1, \vec{k}_2} U(\vec{k}_1 - \vec{k}) g^{(0)}(\vec{k}_1) U(\vec{k} - \vec{k}_1) U(\vec{k}_2 - \vec{k}) g^{(0)}(\vec{k}_2) U(\vec{k} - \vec{k}_2)$$

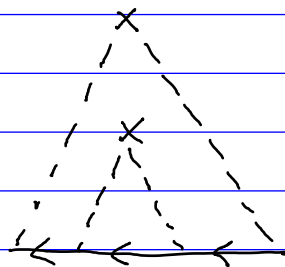
The diagram is reducible, as it falls apart by cutting the middle electron line (the cut is indicated by the red line)

Right diagram:



$$= N^2 [g^{(0)}(\vec{k})]^2 \sum_{\vec{k}_1, \vec{k}_2} U(\vec{k}_1 - \vec{k}) g^{(0)}(\vec{k}_1) U(\vec{k}_2 - \vec{k}_1) g^{(0)}(\vec{k}_2) U(\vec{k}_1 - \vec{k}_2) \cdot g^{(0)}(\vec{k}_1) U(\vec{k} - \vec{k}_1)$$

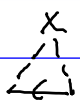
The diagram is irreducible, as it does not fall apart by cutting a single internal electron line. The corresponding self-energy diagram is

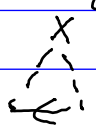


(b) 1. Any term in $\bar{g}$ can be written on the form

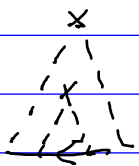
$$g^{(0)} \Sigma^{j_1} g^{(0)} \Sigma^{j_2} g^{(0)} \dots g^{(0)} \Sigma^{j_m} g^{(0)}$$

where Σ^{j_i} is a self-energy diagram, and m is the number of self-energy diagrams in the term. The exact $\bar{g}$ is obtained by summing over m ($m=0.. \infty$), and for each m summing each factor Σ^{j_i} over all self-energy diagrams. Typically one however restricts to some approximative self-energy, corresponding to only summing each Σ^{j_i} over a certain subset of all self-energy diagrams. Even if that subset is finite, one still generates an infinite subset of Feynman diagrams from it, as there is no limit to the number m of self-energy diagrams in a term.

2. In the case of the 1st Born approximation, the only self-energy diagrams taken into account are  and .

The left Feynman diagram is included in this approximation, as it contains two self-energy diagrams of type  which is included in Σ_{1B} .

In contrast, the right Feynman diagram is not included, as it contains the self-energy diagram



which is not included in Σ_{1B} .

(c) The 1st Born approximation for $\bar{g}$ becomes

$$\bar{g}(\vec{k}, ip_m) = \frac{1}{ip_m - \xi_{\vec{k}} - \Delta + \frac{i}{2\tau} \text{sgn}(p_m)}$$

$\bar{G}^R(\vec{k}, \omega)$ is obtained from analytic continuation:

$$\bar{G}^R(\vec{k}, \omega) = \bar{g}(\vec{k}, ip_m) \big|_{ip_m \rightarrow \omega + i\eta}$$

Thus the term ip_m in $\bar{g}$ is replaced by $\omega + i\eta$, while $\text{sgn}(p_m) = \text{sgn}(\text{Im}(ip_m))$ is replaced by $\text{sgn}(\text{Im}(\omega + i\eta)) \underset{\omega \text{ real}}{=} \text{sgn}(\text{Im}(i\eta)) = \text{sgn}(\eta) = +1$. This gives

$$\bar{G}^R(\vec{k}, \omega) = \frac{1}{\omega + i\eta - \xi_{\vec{k}} - \Delta + \frac{i}{2\tau}} \stackrel{\tau \text{ finite}}{\underset{\eta \rightarrow 0}{\approx}} \frac{1}{\omega - (\xi_{\vec{k}} + \Delta) + \frac{i}{2\tau}}$$

where in the last transition we could neglect the infinitesimal $i\eta$ since $\frac{i}{2\tau}$ is finite.

$$\begin{aligned} A(\vec{k}, \omega) &= -\frac{1}{\pi} \text{Im} \bar{G}^R(\vec{k}, \omega) = -\frac{1}{\pi} \text{Im} \frac{\omega - (\xi_{\vec{k}} + \Delta) - \frac{i}{2\tau}}{(\omega - (\xi_{\vec{k}} + \Delta))^2 + \left(\frac{1}{2\tau}\right)^2} \\ &= \frac{1}{\pi} \frac{1/2\tau}{(\omega - (\xi_{\vec{k}} + \Delta))^2 + (1/2\tau)^2} \end{aligned}$$

