

Solution

Exam TFX4210

May 12, 2020

Note: Students were allowed to use lecture notes during the exam.

Problem 1

a) Using the operators and following the steps in lecture notes, we obtain

$$E_k = -\gamma_k - \mu$$

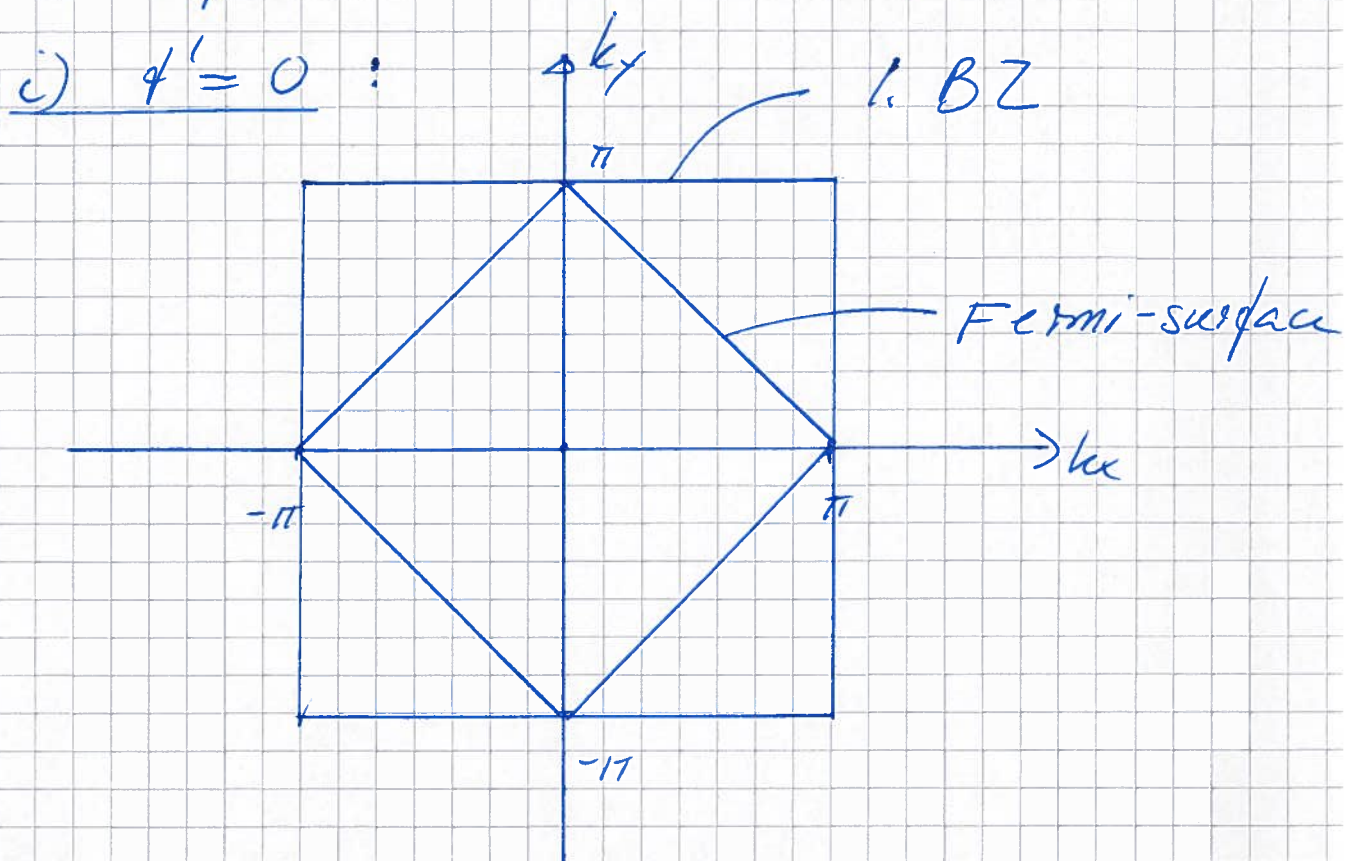
$$\gamma_k = \sum_{\vec{\delta}} \phi(\vec{\delta}) e^{i \vec{k} \cdot \vec{\delta}}$$

$$\vec{\delta} = \vec{r}_i - \vec{r}_j$$

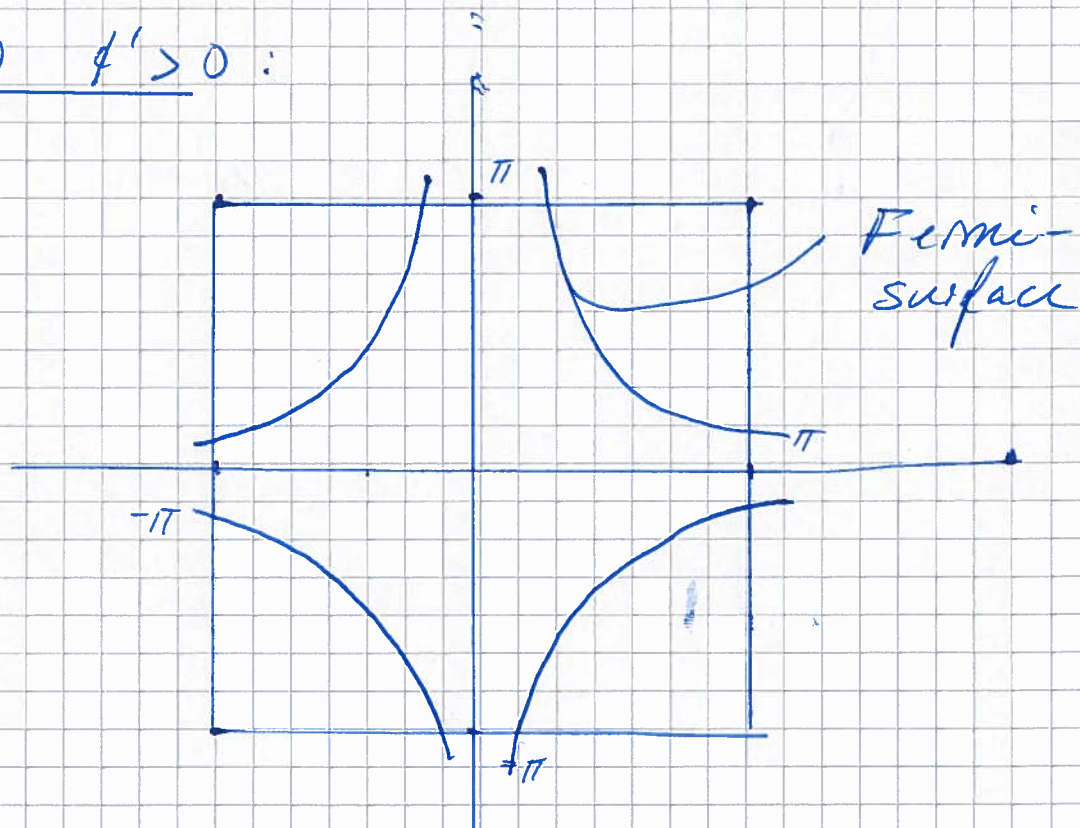
b) For a 2D square lattice, with nearest-neighbor hopping ϕ and next-nearest hopping ϕ' , we find

$$\gamma_k = -2\phi (\cos k_x + \cos k_y) - 4\phi' \cos k_x \cdot \cos k_y$$

c) $\mu = 0$ Fermi-surface



ii) $\phi' > 0$:



d)

$$\frac{\langle N_f \rangle}{N} = \frac{1}{N} \sum_{\vec{k}, \sigma} \langle n_{k\sigma} \rangle$$

$$= \frac{1}{N} \sum_{\vec{k}, \sigma} \frac{1}{e^{\beta E_k} + 1}$$

$$= \frac{2}{N} \sum_{\vec{k}} \frac{1}{e^{\beta E_k} + 1}$$

$$E_k = -\epsilon_k - \mu$$

Oppgave 2

The model is the same as treated in lectures, with an additional anisotropy term

$$- K \sum_i S_{iz}^2$$

$$J < 0 ; K > 0$$

The spectrum has been found by detailed calculations with $K=0$ in class. In that case, we

found

$$H = \tilde{E}_0 + \sum_q \omega_q \left(A_q^\dagger A_q + \frac{1}{2} + B_q^\dagger B_q + \frac{1}{2} \right)$$

$$\omega_q = 2|J|S \sqrt{z^2 - \gamma_q^2}$$

$z = \#$ number of nearest neighbors

$$\gamma(q) = \sum_{\vec{s}} e^{i\vec{q} \cdot \vec{s}}$$

$$\tilde{E}_0 = 2N J z S(S+1)$$

$N = \#$ lattice sites on each sub-lattice

a) Here, the candidates are supposed to realize that by following the procedure for solving this problem used in class (for $K=0$), one expands to second order in boson-operators via the Holstein-Primakoff transformation.

At one intermediate stage, one has

$$H = E_0 - 2JSz \sum_{\vec{q}} (a_{\vec{q}}^{\dagger} a_{\vec{q}} + b_{\vec{q}}^{\dagger} b_{\vec{q}}) \\ - 2JS \sum_{\vec{q}} \gamma(\vec{q}) (a_{\vec{q}} b_{\vec{q}} + b_{\vec{q}}^{\dagger} a_{\vec{q}}^{\dagger})$$

By using the same transformation on the anisotropy-term keeping terms to quadratic, this modifies $\tilde{E}_0$, and also leads to an additional term like the second term above. One may view this as a shift of the parameter z .

The additional term like the second, is

$$2KS \sum_{\vec{q}} (a_{\vec{q}}^{\dagger} a_{\vec{q}} + b_{\vec{q}}^{\dagger} b_{\vec{q}})$$

which we may add to the second term, giving

$$\begin{aligned} & - (2JSz - 2KS) \sum_{\vec{q}} (a_{\vec{q}}^{\dagger} a_{\vec{q}} + b_{\vec{q}}^{\dagger} b_{\vec{q}}) \\ & = - 2JSz' \sum_{\vec{q}} (a_{\vec{q}}^{\dagger} a_{\vec{q}} + b_{\vec{q}}^{\dagger} b_{\vec{q}}) \end{aligned}$$

$z' = z + \frac{K}{J}$

The Hamiltonian is now in precisely the same form as for $K=0$, and we obtain $\omega_{\vec{q}}$ by copying the result from lecture notes

$$\omega_{\vec{q}} = 2|J|S \sqrt{\left(z + \frac{K}{J}\right)^2 - \gamma_{\vec{q}}^2} \quad \vec{q} \in \text{reduced Brillouin zone}$$

b)

For $K=0$ this has been done in lecture notes.

Considering magnetization along z -direction on one sublattice, we have

$$M = \frac{1}{N} \left\langle \sum_{i \in A} S_{iz} \right\rangle_A$$

where A denotes one sublattice
and where M is the magnetization
per lattice site

$$M = S - \frac{1}{N} \sum_q \langle a_q^\dagger a_q \rangle$$

where q runs over a reduced
Brillouin-zone. The above average
is taken with respect to the
diagonalized Hamiltonian.

$$a_q = u_q A_q - v_q B_q^\dagger$$

$$a_q^\dagger = u_q A_q^\dagger - v_q B_q$$

We thus find

$$\begin{aligned} \langle a_q^\dagger a_q \rangle &= u_q^2 \langle A_q^\dagger A_q \rangle \\ &\quad + v_q^2 \langle B_q B_q^\dagger \rangle \\ &= u_q^2 \langle A_q^\dagger A_q \rangle + v_q^2 [\langle B_q^\dagger B_q \rangle + 1] \end{aligned}$$

$$M = S - \frac{1}{N} \sum_q v_q^2 - \frac{1}{N} \sum_q \left(\frac{u_q^2 + v_q^2}{e^{\beta \omega_q} - 1} \right)$$

Low-damp corrections:

The term involving the Bose-distribution is exponentially suppressed as $T \rightarrow 0$, so this term is negligible. Thus, we have

$$M = S - \frac{1}{N} \sum_q v_q^2$$

Only quantum-fluctuations are left.

c)

From lecture notes:

$$u_g^2 - v_g^2 = 1$$

$$u_g^2 = \frac{1}{2} \left(1 + \frac{z'}{|z'|^2 - \gamma_g^2} \right)$$

$$v_g^2 = \frac{1}{2} \left(-1 + \frac{z'}{|z'|^2 - \gamma_g^2} \right)$$

$$z' = z + \frac{K}{|j|}$$

$$\lim_{g \rightarrow 0} \gamma_g = z$$

$$\omega_g = 2|j|S \sqrt{|z'|^2 - \gamma_g^2}$$

ω_g : Finite when $g \rightarrow 0$, $K \neq 0$!

Thermal fluctuations are completely suppressed at low T due to the gap in ω_g as $g \rightarrow 0$.

Quantum-fluctuations are also small, since v_g^2 is also finite when $g \rightarrow 0$.