

Solution Exam TFY4210/FY8302
May 26, 2021

1a)
$$H = -J \sum_{\langle i,j \rangle} [\sigma_{iz} \sigma_{jz} + \Delta (\sigma_{ix} \sigma_{jx} + \sigma_{iy} \sigma_{jy})]$$
$$= -J \sum_{\langle i,j \rangle} (\sigma_{iz} \sigma_{jz} + \Delta \sigma_{i+} \sigma_{j-})$$

Now using the Holstein-Primakoff transformation, computing to quadratic order, we find

$$H = E_0 - J \sum_{\langle i,j \rangle} 2S [a_i^\dagger a_i + b_i^\dagger b_i + \Delta (a_i b_j + b_i^\dagger a_j^\dagger)]$$

Now, introducing Fourier-transformed operators, we may write

$$H = E_0 - 2JS \sum_{\mathbf{q}} (a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + b_{\mathbf{q}}^\dagger b_{\mathbf{q}}) - 2JS \sum_{\mathbf{q}} \Delta \gamma(\mathbf{q}) (a_{\mathbf{q}} b_{\mathbf{q}} + b_{\mathbf{q}}^\dagger a_{\mathbf{q}}^\dagger)$$
$$\gamma(\mathbf{q}) = \sum_{\mathbf{\delta}} e^{i \mathbf{q} \cdot \mathbf{\delta}}$$

$\mathbf{\delta}$: Vector connecting i to its nearest neighbor.

If we now define

$$\tilde{\gamma}(\vec{q}) = \Delta \gamma(\vec{q}),$$

then H has exactly the same form as the $\Delta=1$ case, and we can immediately write

$$H = 2NJz S(S+1)$$

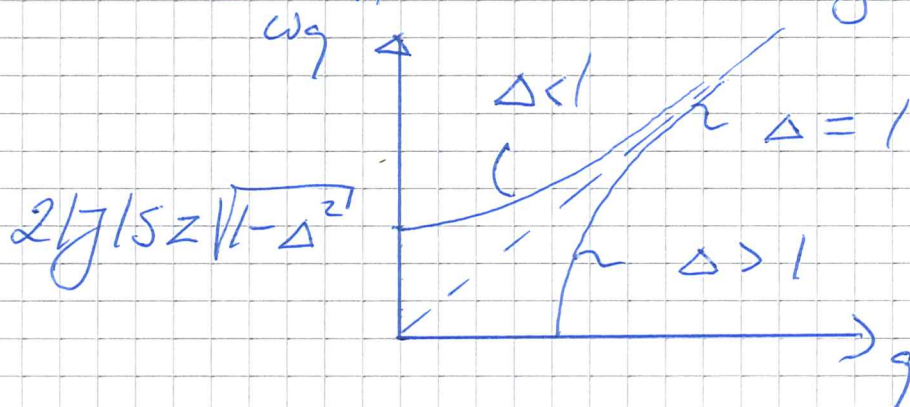
$$+ \sum_{\vec{q}} \hbar \omega_{\vec{q}} (A_{\vec{q}}^{\dagger} A_{\vec{q}} + 1/2)$$

$$+ \sum_{\vec{q}} \hbar \omega_{\vec{q}} (B_{\vec{q}}^{\dagger} B_{\vec{q}} + 1/2)$$

$$\omega_{\vec{q}} = 2J/S \sqrt{z^2 - \tilde{\gamma}^2(\vec{q})}$$

$$= 2J/S \left(z^2 - \Delta^2 \gamma^2(\vec{q}) \right)^{1/2}$$

$z = \#$ nearest neighbors



1b) From the lecture notes, we have

$$M = 5 - \frac{1}{N} \sum_q V_q^2 - \frac{1}{N} \sum_q \frac{u_q^2 + v_q^2}{e^{\beta \omega_q} - 1}$$

The temperature-corrections are in the third term.

At low T , the integral is completely dominated by small momenta, since the factor $e^{-\beta \omega_q}$ serves as an efficient cutoff-factor.

$$u_q^2 + v_q^2 = \frac{z}{\sqrt{z^2 - \Delta^2 \gamma^2}} \quad ; \quad \Delta < 1$$

$$C(T) \equiv \frac{1}{N} \sum_q \frac{u_q^2 + v_q^2}{e^{\beta \omega_q} - 1} \quad \text{Thermal correction to } M$$

$$C(T) = \frac{1}{N} \sum_q e^{-\beta \omega_q} \frac{z}{\sqrt{z^2 - \Delta^2 \gamma^2}} \sum_{n=0}^{\infty} e^{-n \beta \omega_q}$$

Since $\lim_{q \rightarrow 0} \omega_q \neq 0 \quad ; \quad \Delta < 1$

the integral is completely dominated

by the term $n=0$, $\beta|J| \gg 1$. Thus, we get

$$C(T) \approx \frac{1}{N} \sum_g \frac{z}{(z^2 - \Delta^2 \gamma^2)^{1/2}} e^{-\beta \omega_g}$$

$$\beta \omega_g = \eta (z^2 - \Delta^2 \gamma^2)^{1/2}; \quad \eta = 2S\beta|J|$$

$$\text{For small } g: z^2 - \Delta^2 \gamma^2 \approx z^2 - \Delta^2 (z - \frac{1}{2}g^2)^2$$

$$= z^2(1 - \Delta^2) + \Delta^2 z g^2 = a + b g^2$$

$$a \equiv z^2(1 - \Delta^2); \quad b \equiv \Delta^2 z$$

Converting the g -sum to an integral:

$$C(T) = \frac{1}{2\pi} z \int_0^\infty \frac{dg g}{\sqrt{a + b g^2}} e^{-\eta \sqrt{a + b g^2}}$$

$$= \frac{z}{4\pi} \int_0^\infty \frac{dx}{\sqrt{a + b x}} e^{-\eta \sqrt{a + b x}}$$

$$= \frac{z}{4\pi} \frac{2}{b\eta} e^{-\eta \sqrt{a}}$$

$$= \frac{z \sqrt{1 - \Delta^2}}{2S\beta|J|}$$

$$C(T) = \frac{1}{4\pi} \frac{1}{\Delta^2 S} \frac{k_B T}{|J|} e$$

Note the drastic suppression of $C(T)$

with increasing z and S , at

a given low temperature $k_B T \ll |J|$.

When $\Delta = 1$, the thermal corrections to M is a power-law in (T/J) , $\sim (T/J)^{d-1}$, $d > 2$

When $\Delta < 1$, the corrections are exponentially small in J/T .

This drastic change is due to the gap in the spin-wave spectrum as $q \rightarrow 0$ when $\Delta < 1$.

1c) We assume to begin with spin-ordering along z -axis in spin-space. When $\Delta < 1$ this makes sense, since it is more energetically favorable to order spins along z -axis than along (x, y) -axes in spin space.

This is reflected in the fact that when $q \rightarrow 0$, $\omega_q \neq 0$ for $\Delta < 1$.

This is the typical situation in an Ising-model, which does not have low-energy spin-waves.

When $\Delta > 1$, it is energetically favorable to order spins along (x, y) -directions in spin-space, Thus the z -axis ordering presumed in the H-P-transformation makes little sense.

For $\Delta > 1$

$$\omega_q = 2|J|S (z^2 - \Delta^2 \gamma^2)^{1/2}$$

becomes imaginary for low q .

This reflects an instability from a spin-state ordered along z -axis to a spin-state ordered in the (x, y) -plane in spin-space. Our starting point with spin-ordering along z -axis + small assumed fluctuations (H-P-transformation) is just not the correct starting point. We should have assumed spin-ordering in the (x, y) -plane and considered a modified spin-wave theory around such a starting point.

Problem 2

- a) i) Scattering of electrons on opposite sides of the Fermi-surface $(\mathbf{k}, -\mathbf{k})$ to $(\mathbf{k}', -\mathbf{k}')$ within a thin energy shell around Fermi-surface
- ii) Electrons that interact are in opposite spin-states, such that the electron-pair forms a spin-singlet.

b) The BCS gap-equation:

$$\Delta(\mathbf{k}) = - \sum_{\mathbf{k}'} V_{\mathbf{k}, \mathbf{k}'} \Delta(\mathbf{k}') \chi_{\mathbf{k}'}$$

$$\chi_{\mathbf{k}'} = \frac{1}{2E_{\mathbf{k}'}} \tanh\left(\frac{\beta E_{\mathbf{k}'}}{2}\right)$$

$$E_{\mathbf{k}'} = \left((\epsilon_{\mathbf{k}'} - \mu)^2 + |\Delta(\mathbf{k}')|^2 \right)^{1/2}$$

$$\beta = \frac{1}{k_B T}$$

c) For this particular form of $V_{\mathbf{k}, \mathbf{k}'}$:

$$\Delta(k) = V g(k) \underbrace{\sum_{k'} g(k') \Delta(k') \chi_{k'}}_{\vec{k}\text{-independent}}$$

Thus, we have

$$\Delta(\vec{k}) = \Delta_0 g(\vec{k})$$

$$\underline{\underline{F(\vec{k}) = g(\vec{k})}}$$

$$\Delta_0 \equiv V \sum_{k'} g(k') \underbrace{\Delta(k')}_{= \Delta_0 g(k')} \chi_{k'}$$

$$\Delta_0 = V \Delta_0 \sum_{k'} (g(k'))^2 \chi_{k'}$$

Equation determining Δ_0 :

$$1 = V \sum_{k'} (g(k'))^2 \chi_{k'}$$

$$\underline{\underline{E_{k'} = \left((\epsilon_{k'} - \mu)^2 + \Delta_0^2 (g(k'))^2 \right)^{1/2}}}$$

$$d) \quad \Delta(\vec{k}) = \Delta_0(T) g(\vec{k})$$

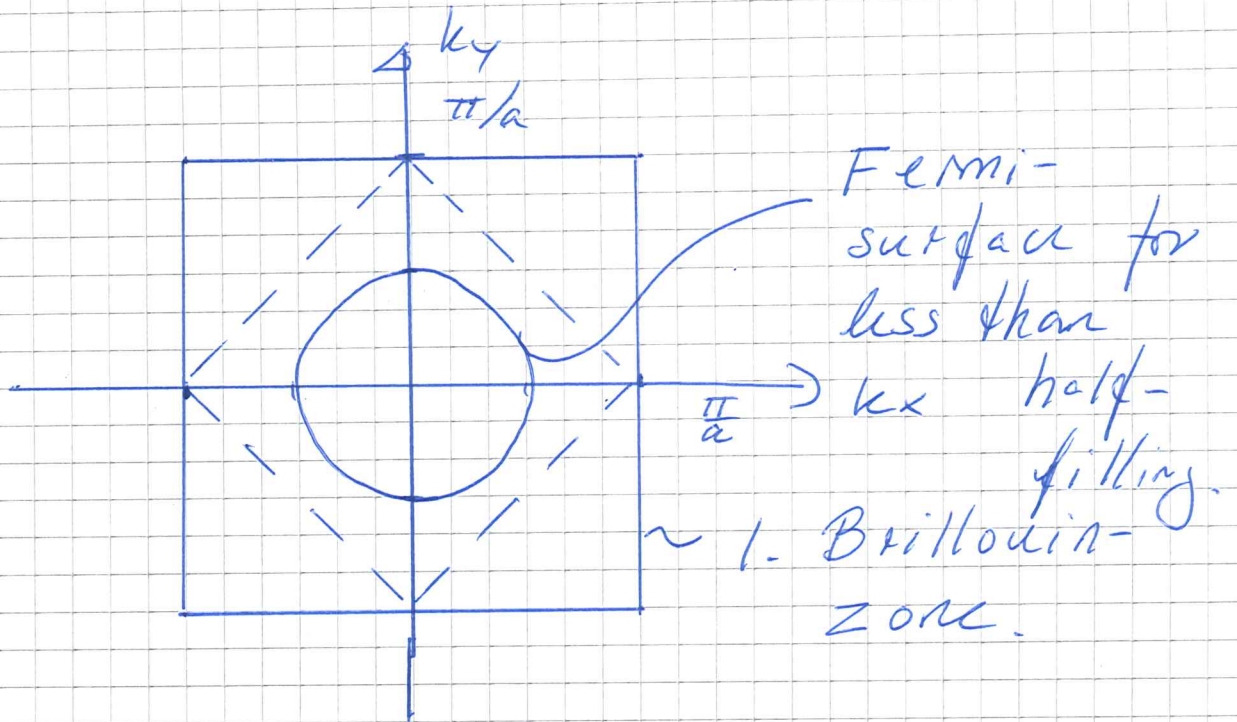
In addition, the pairing amplitude $b_k \equiv \langle C_{k\uparrow}^\dagger C_{-k\downarrow}^\dagger \rangle$

is spin-singlet. This state is antisymmetric under interchange of two electrons, and the spin-part of the pairing state is antisymmetric (spin singlet)

Therefore, the $\vec{k}$ -dependent part of the pairing state must be symmetric under interchange of two electrons, i.e. symmetric when $\vec{k} \rightarrow -\vec{k}$.

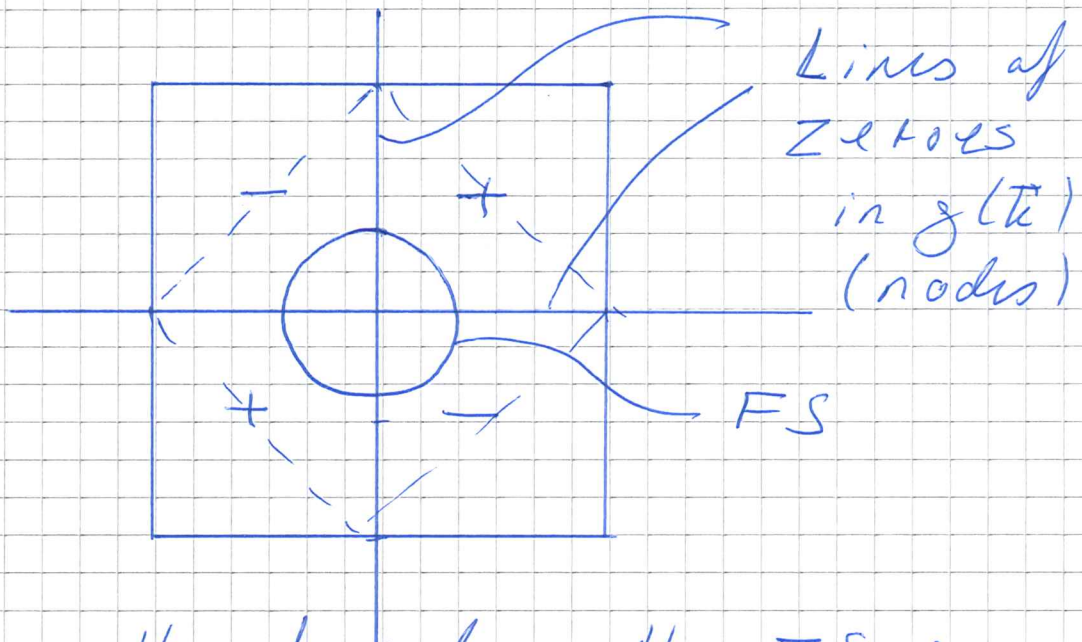
The first two functions $g(\vec{k})$ are odd when $\vec{k} \rightarrow -\vec{k}$, and are therefore not permissible. The other 3 are permissible

e)



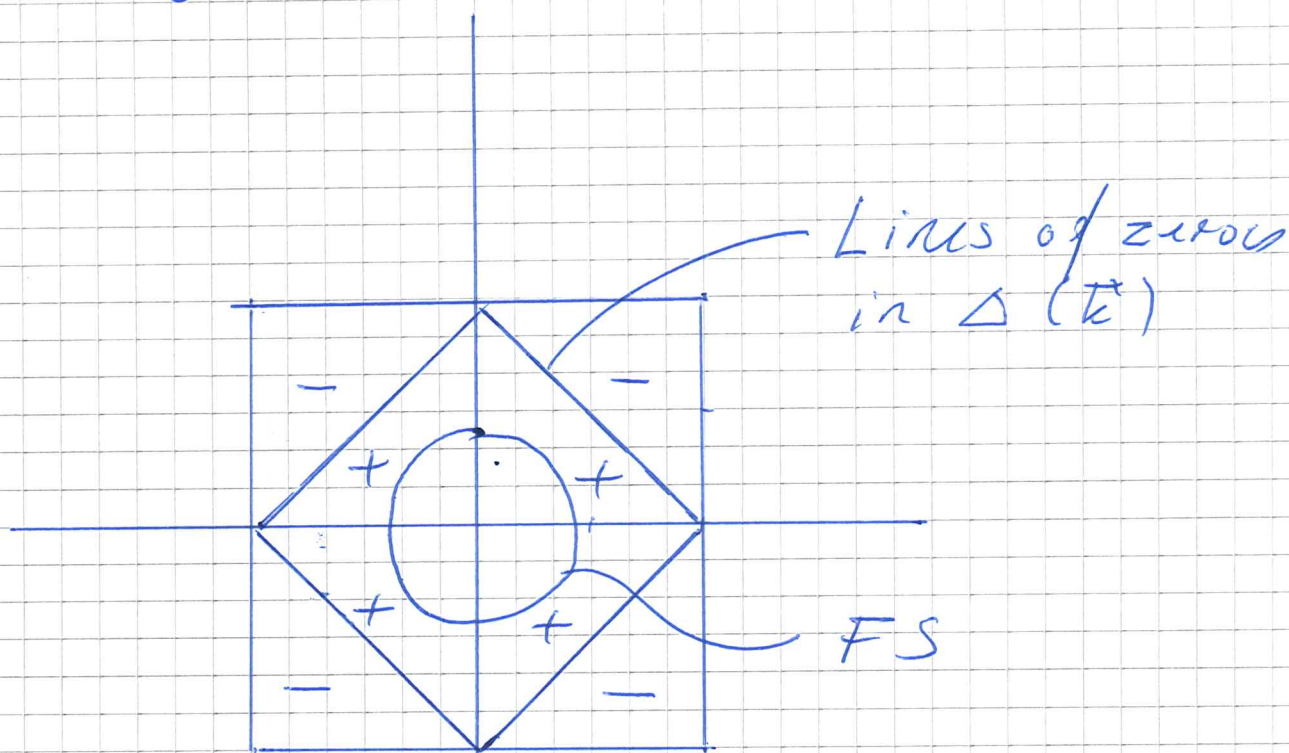
Take the three permissible functions and consider their signs on the 1. B-zone:

i) $g(\vec{k}) = \sin k_x \sin k_y$



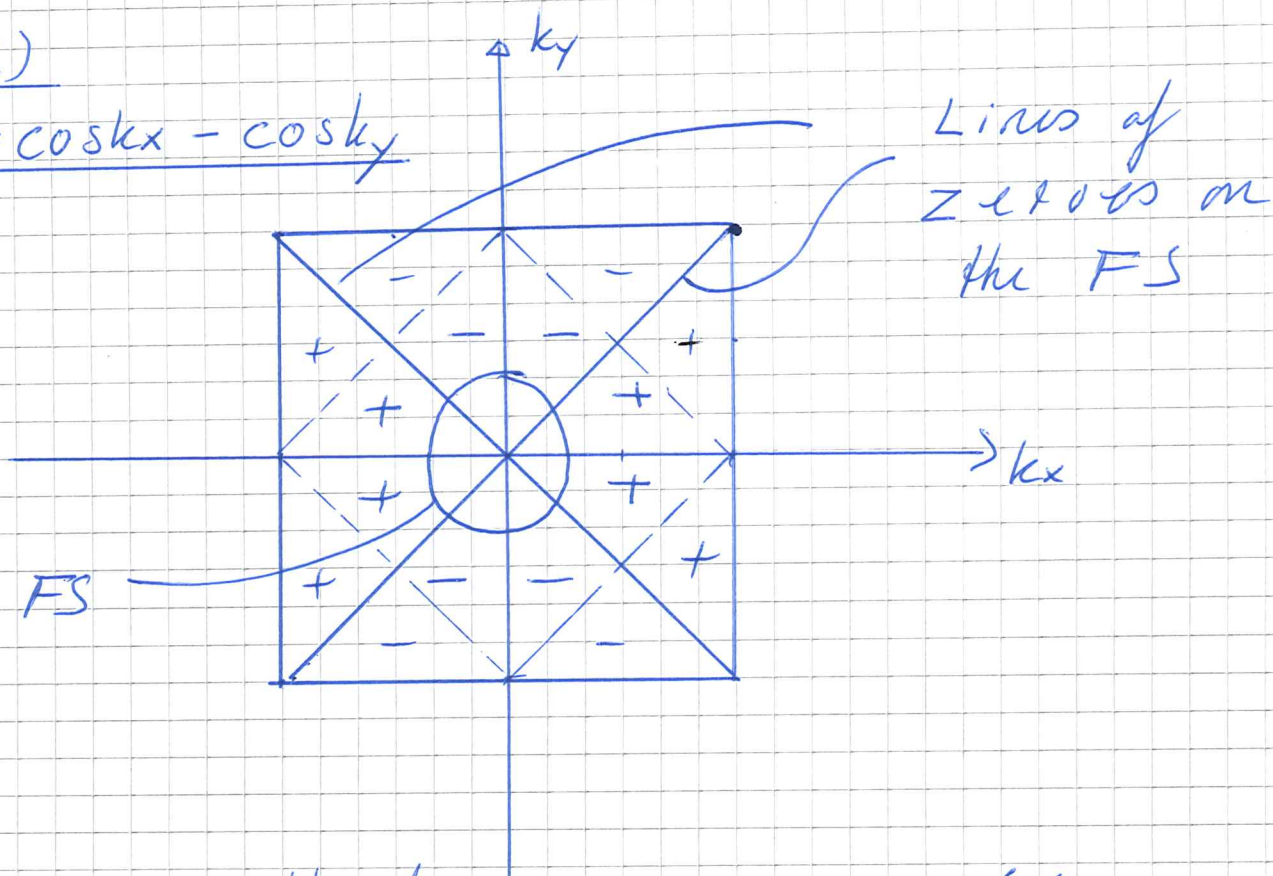
No matter how large the FS is (for less than half-filling) the gap changes sign four times on the FS.

ii) $\underline{g(\vec{k}) = \cos k_x + \cos k_y}$



This gap will change sign on the Fermi-surface. This is just an anisotropic version of the $\vec{k}$ -independent "s-wave" gap we find when $g(\vec{k}) = 1$. It is often referred to as "anisotropic s-wave gap".

cc) $g(k) = \cos k_x - \cos k_y$



No matter where the FS is (less than half-filling) the gap always changes sign four times on the FS.

This form of the gap is believed to be the essentially correct form of the gap-function $\Delta(k)$ in high- T_c cuprates.