1

Exam TFY4210/FY8302
May 27, 2024

Problem 1

a) First term: Hopping of single electron from site j to site i, in spin-state σ.

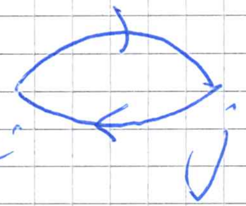
Second term: A term $-\mu N$ where μ is chemical potential and $N = \sum_{i\sigma} n_{i\sigma}$. μ regulates the average # particles on the lattice

Third term: On-site Coulomb-repulsion term. It is operative only if two electrons occupy the same site, and due to the Pauli-principle, these two electrons must have different spins.

(2)

b) $$H = -\sum_{\langle i,j\rangle \alpha} J_\alpha S_{i\alpha} S_{j\alpha}$$

Such an effective low-energy Hamiltonian may be derived from the Hubbard model when the lattice is half-filled (one fermion pr. site), and $t \ll U$.

Virtual hopping 

lowers kinetic energy and produces an exchange term

$$J \sim -\frac{t^2}{U}$$

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c) $$\sum_{\langle i,j\rangle} a_i^+ a_i$$

$$= z \sum_i a_i^+ a_i = \frac{z}{N} \sum_{q_1 q_2} \sum_i a_{q_1}^+ a_{q_2} e^{i\bar{r}_i \cdot (\bar{q}_2 - \bar{q}_1)}$$

$$= z \sum_q a_q^+ a_q$$

<u>Similarly:</u>

$$\sum_{\langle i,j\rangle} b_i^+ b_i = z \sum_q b_q^+ b_q$$

$$\sum_{\langle i,j\rangle} a_i b_j$$

$$= \frac{1}{N} \sum_{i,\vec{\delta}} \sum_{\vec{q}_1,\vec{q}_2} a_{q_1} b_{q_2} e^{i(\vec{q}_1-\vec{q}_2)\cdot\vec{r}_i - i\vec{q}_2\cdot\vec{\delta}}$$

$$= \sum_q a_q b_q \underbrace{\sum_{\vec{\delta}} e^{-i\vec{q}\cdot\vec{\delta}}}_{= 2(\cos q_x + \cos q_y)}$$

on a 2D square lattice

$$= \sum_q \gamma(\vec{q})\, a_q b_q$$

where $\gamma(\vec{q}) = 2(\cos q_x + \cos q_y)$

<u>Similarly:</u>

$$\sum_{\langle i,j\rangle} a_i^+ b_j^+ = \sum_q \gamma(\vec{q})\, a_q^+ b_q^+$$

<u>Collecting terms:</u>

$$H = E_0^c - 2JS \sum_q \left[z\left(a_q^+ a_q + b_q^+ b_q\right) + \Delta\,\gamma(q)\left(a_q b_q + a_q^+ b_q^+\right) \right]$$

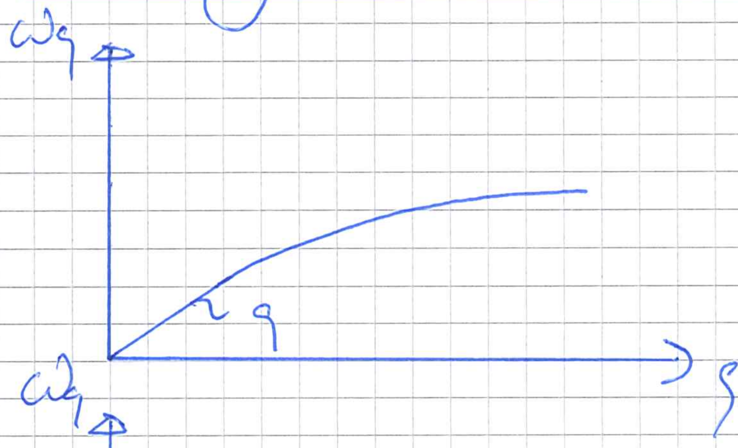
q. e. d.

<u>d)</u> By redefining $\Delta\,\gamma(q) = \tilde{\gamma}(q)$ we just follow exactly the same steps as for $\Delta = 1$, to obtain

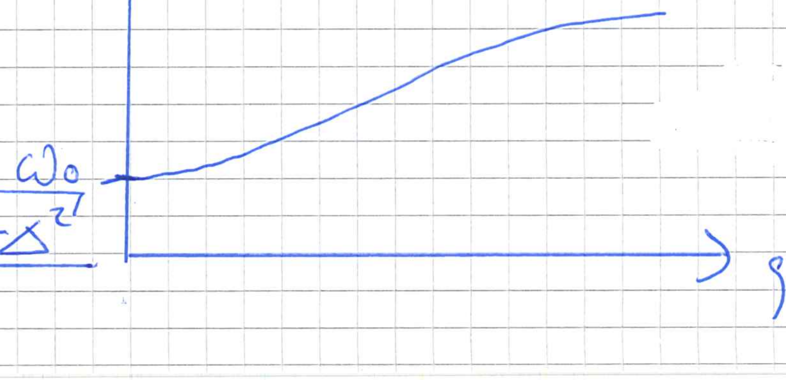
$$\omega_q = 2|J|S\sqrt{z^2 - \tilde{\gamma}^2(q)}$$

$$= 2|J|S\sqrt{z^2 - \Delta^2\gamma^2(q)}$$

$\Delta = 1$:



$\Delta < 1$



$\omega_0 = 2|J|Sz\sqrt{1-\Delta^2}$

e) <u>Zeeman-term</u>

$$h\sum_q \left(a_q^+ a_q - b_q^+ b_q\right)$$

Use Bogoliubov-transformation

$$\left(u_q A_q^+ - v_q B_q\right)\left(u_q A_q - v_q B_q^+\right)$$

$$-\left(u_q B_q^+ - v_q A_q\right)\left(u_q B_q - v_q A_q^+\right)$$

$$= u_q^2 A_q^+ A_q + v_q^2 B_q B_q^+ - v_q u_q B_q A_q - v_q u_q A_q^+ B_q^+$$

$$-\left(u_q^2 B_q^+ B_q + v_q^2 A_q A_q^+ - v_q u_q A_q B_q - v_q u_q A_q^+ B_q^+\right)$$

$$= \left(u_q^2 - v_q^2\right) A_q^+ A_q - v_q^2$$

$$- \left(u_q^2 - v_q^2\right) B_q^+ B_q + v_q^2$$

$$= A_q^+ A_q - B_q^+ B_q \qquad , u_q^2 - v_q^2 = 1$$

$$H = E_0^{qM} + \sum_q \omega_q \left(A_q^+ A_q + B_q^+ B_q\right)$$

$$+ h\sum_q \left(A_q^+ A_q - B_q^+ B_q\right)$$

$$= E_0^{gM} + \sum_q \left(\omega_q^A A_q^+ A_q + \omega_q^B B_q^+ B_q \right)$$

$$\omega_q^A = \omega_q + h$$

$$\omega_q^B = \omega_q - h$$

The two degenerate magnon bands are split by the Zeeman-term.

f) $\min(q)\ \omega_q = 2|J|Sz\sqrt{1-\Delta^2}$

$h>0$: ω_q^B may change sign.

The lowest field at which this happens, is when

$$2|J|Sz\sqrt{1-\Delta^2} - h = 0$$

$$h = 2|J|Sz\sqrt{1-\Delta^2}$$

If $h<0$, then ω_q^A may change sign.

Physical interpretation:

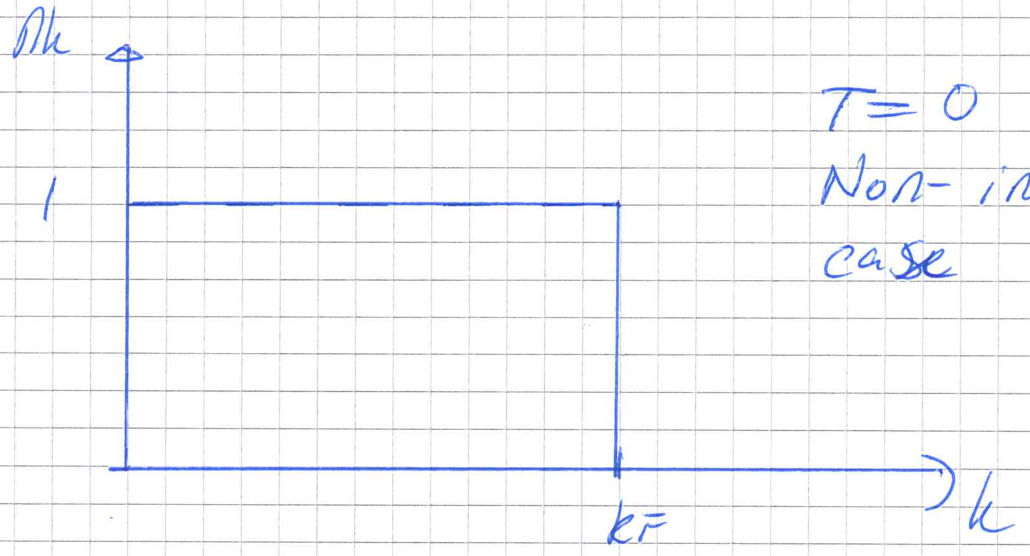
The external uniform field tries to make the spins point parallell to the field. However, ω_q^A, ω_q^B are derived under the assumption that the spins are almost anti-parallell (modulo small magnon-fluctuations).

For large enough h, the spins are forced to be instead parallell, and the negative value of ω_q^A or ω_q^B signals that the nearly anti-parallell spin-ordering is no longer true.

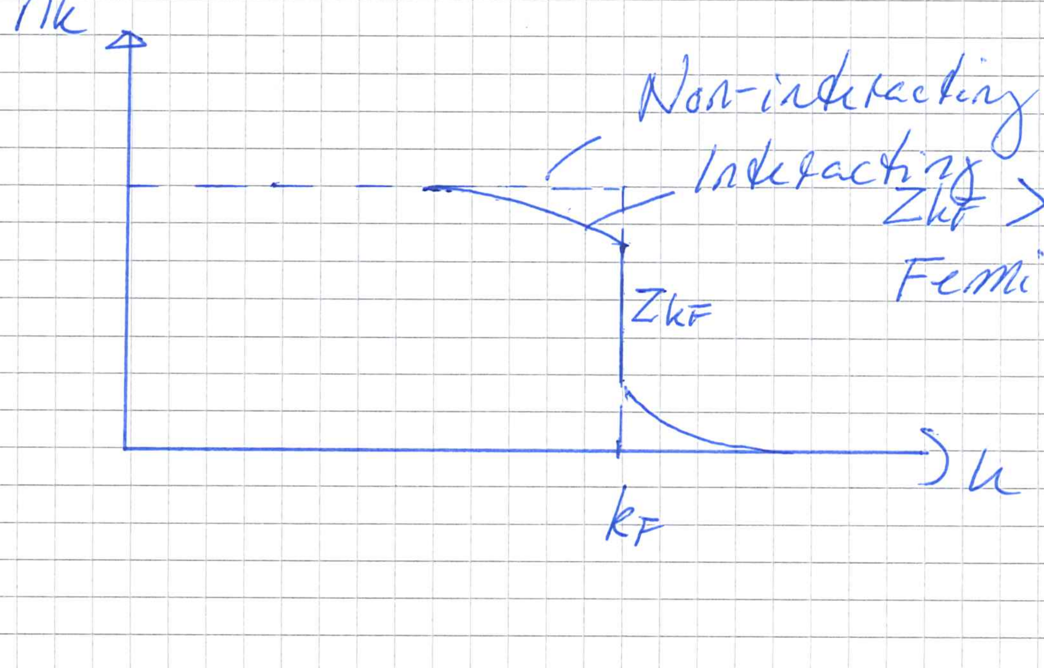
Problem 2

a) A Fermi-liquid is an interacting fermion systems where the quantum states $(\vec{k},\sigma)$ are in a one-to-one correspondence with the $(\vec{k},\sigma)$-states of the non-interacting system.

Momentum distribution n_k:



$T=0$
Non-interacting case



$Z_{k_F} > 0$: Fermi liquid.

9

b) Interaction term:

$$-|U|\sum_i c_{i\uparrow}^{+} c_{i\uparrow} c_{i\downarrow}^{+} c_{i\downarrow}.$$

$$c_{i\uparrow}^{+} c_{i\uparrow} c_{i\downarrow}^{+} c_{i\downarrow}$$

$$= c_{i\uparrow}^{+} c_{i\downarrow}^{+} c_{i\downarrow} c_{i\uparrow}$$

If we identify $P_i \equiv c_{i\downarrow} c_{i\uparrow}$

then we have that

$$c_{i\uparrow}^{+} c_{i\downarrow}^{+} = P_i^{+}$$

$$P_i = c_{i\downarrow} c_{i\uparrow}$$

$$\underline{\underline{P_i^{+} = c_{i\uparrow}^{+} c_{i\downarrow}^{+}}}$$

c)

$$P_i^{+} P_i$$

$$= (b_i^{+} + \delta b_i^{+})(b_i + \delta b_i)$$

$$= b_i^{+} b_i + b_i^{+} (P_i - \cancel{b_i})$$

$$+ (P_i^{+} - b_i^{+}) b_i + \mathcal{O}(\delta b_i^{+} \delta b_i)$$

$$\simeq b_i^{+} P_i + P_i^{+} b_i - b_i^{+} b_i$$

$$H = -t\sum_{\langle ij\rangle\sigma} c_{i\sigma}^{+} c_{j\sigma} - \mu \sum_{i\sigma} n_{i\sigma}$$

$$+ U\sum_i b_i^{+} b_i - |U| \sum_i \left(b_i^{+} P_i + b_i P_i^{+} \right)$$

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d) $b_i^{+} = b_i = b$

$$U \sum_i b_i^{+} b_i = U N b^2$$

$$|U| \sum_i \left(b_i^{+} P_i + b_i P_i^{+} \right)$$

$$= |U| b \sum_i \left(P_i^{+} + P_i \right)$$

$$= |U| b \sum_k \left(C_{k\uparrow}^{+} C_{-k\downarrow}^{+} + C_{-k\downarrow} C_{k\uparrow} \right)$$

Define $\Delta \equiv |U| b$

$$|U| N b^2 = N \frac{\Delta^2}{|U|}$$

$$-t\sum_{\langle ij\rangle\sigma} c_{i\sigma}^{+} c_{j\sigma} - \mu \sum_{i\sigma} n_{i\sigma}$$

$$= \sum_{k\sigma} \varepsilon_k \, C_{k\sigma}^{+} C_{k\sigma} \; ; \; \underline{\underline{\varepsilon_k = -t \sum_{\delta} e^{i\vec{k}\cdot\vec{\delta}} - \mu}}$$

$\{\delta\}$: Set of vectors connecting site i to j where ij are nearest neighbors.

$$H = \sum_{k\sigma} \varepsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + \frac{N\Delta^2}{|U|} - \Delta \sum_{k} \left(c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow} \right)$$

$$\varepsilon_k = -t \sum_{\delta} e^{i k \cdot \delta} - \mu$$

$$\Delta = |U|\, b$$

e)

$$n_k = \sum \langle c^{\dagger}_{k\sigma} c_{k\sigma} \rangle$$

$$= \langle c^{\dagger}_{k\uparrow} c_{k\uparrow} + c^{\dagger}_{k\downarrow} c_{k\downarrow} \rangle$$

$$= \Big[u_k^2 \langle \eta^{\dagger}_k \eta_k \rangle + v_k^2 \langle \gamma^{\dagger}_k \gamma_k \rangle + u_k^2 \langle \gamma_{-k} \gamma^{\dagger}_{-k} \rangle + v_k^2 \langle \eta_{-k} \eta^{\dagger}_{-k} \rangle \Big]$$

$$= \Big[u_k^2 + v_k^2 - u_k^2 \langle \gamma^{\dagger}_k \gamma_k \rangle - v_k^2 \langle \eta^{\dagger}_k \eta_k \rangle + u_k^2 \langle \eta^{\dagger}_k \eta_k \rangle + v_k^2 \langle \gamma^{\dagger}_k \gamma_k \rangle \Big]$$

$$n_k = 1 + (u_k^2 - v_k^2)\left(\langle \eta_k^\dagger \eta_k \rangle - \langle \gamma_k^\dagger \gamma_k \rangle\right)$$

$$= 1 - \frac{\varepsilon_k}{E_k}\left(\frac{1}{1+e^{-\beta E_k}} - \frac{1}{1+e^{+\beta E_k}}\right)$$

$$= 1 - \frac{\varepsilon_k}{E_k}\, \frac{\left(e^{+\frac{\beta E_k}{2}} - e^{-\frac{\beta E_k}{2}}\right)}{e^{\frac{\beta E_k}{2}} + e^{-\frac{\beta E_k}{2}}}$$

$$= 1 - \frac{\varepsilon_k}{E_k} \tanh\left(\frac{\beta E_k}{2}\right)$$

$T \to 0$; $\beta \to \infty$ $\quad \tanh\left(\frac{\beta E_k}{2}\right) \to 1$

$$n_k = 1 - \frac{\varepsilon_k}{E_k}$$

No discontinuity on the Fermi-surface if $\Delta \neq 0 \Rightarrow$

<u>Not a Fermi-liquid</u>

$\not{b}$ This is a spin-singlet superconductor with a momentum-independent (s-wave) energy gap on the Fermi-surface.