



Contact during the exam:
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Exam in TFY4230 STATISTICAL PHYSICS

Wednesday december 1, 2010
15:00–19:00

Allowed help: Alternativ C

Standard calculator (according to list prepared by NTNU).

K. Rottman: *Matematisk formelsamling* (all languages).

Barnett & Cronin: Mathematical Formulae

This problem set consists of 2 pages.

Problem 1. Particles in a spherical volume

A system of N classical non-relativistic particles is confined to a spherical (3-dimensional) volume with “soft” walls, described by the Hamiltonian

$$H = \sum_{i=1}^N \frac{1}{2m} \mathbf{p}_i^2 + \varepsilon_0 \left(\frac{\mathbf{x}_i^2}{r_0^2} \right)^n, \quad (1)$$

where ε_0 is a positive constant, r_0 is a length characterizing the radius of the sphere, and n is a positive integer.

- Write down the canonical partition function Z for this system at temperature T .
- Calculate the internal energy $U = \langle H \rangle$ and heat capacity C for this system.
- Does your result for C agree with the equipartition theorem when $n = 1$ or $n = \infty$?
- Calculate the mean particle density, defined as

$$\rho(\mathbf{x}) = \left\langle \sum_{i=1}^N \delta(\mathbf{x} - \mathbf{x}_i) \right\rangle. \quad (2)$$

Next assume the particles to have charge Q measured in units of the positron charge e , and that the system is exposed to a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. This implies that we must make the substitution

$$\mathbf{p}_i \rightarrow \mathbf{p}_i + Qe\mathbf{A}(\mathbf{x}_i) \quad (3)$$

in the Hamiltonian (1).

- What is the effect of this magnetic field on the classical partition function Z ?

The Gamma function:

$$\Gamma(\nu) = \int_0^\infty \frac{dt}{t} t^\nu e^{-t}, \quad \Gamma(1 + \nu) = \nu \Gamma(\nu), \quad (4)$$

$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(\nu) = \nu^{-1} + \dots \text{ when } \nu \rightarrow 0. \quad (5)$$

Problem 2. Monte-Carlo simulation of a thermal system

Here you should to prepare for a numerical simulation of the system discussed in the previous problem, for the case of $N = 1$ and $n = 2$. We further simplify the system to be one-dimensional.

- Write down the classical equations of motion dictated by the Hamiltonian (1).
- Find suitable units for time and length so that the equations of motion can be written in terms of dimensionless variables.
- How would you discretize the differential equations for a numerical solution of the problem?
- To simulate temperature one has to introduce additional fluctuating and a damping forces. Indicate how this should be done.

Hamilton's equations:

$$\dot{x}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial x_\alpha}. \quad (6)$$

Problem 3. Quantum statistics of thermal radiation

The eigen-energies for the free radiation field can be written

$$E = \sum_{\mathbf{k}, r} \hbar \omega_{\mathbf{k}} N(\mathbf{k}, r) \quad (7)$$

where $\omega_{\mathbf{k}} = c|\mathbf{k}|$, and where $N(\mathbf{k}, r) = 0, 1, \dots$ is the *occupation number* of the state with wavevector $\mathbf{k}$ and polarization r . We have subtracted the zero-point energy. With av volume V and periodic boundary conditions the allowed values for $\mathbf{k}$ lie on a lattice,

$$\mathbf{k} = \frac{2\pi}{V^{1/3}} (n_x, n_y, n_z) \quad \text{with all } n\text{'s integer.} \quad (8)$$

- Show that the partition function for this system can be written

$$\ln Z = - \sum_{\mathbf{k}, r} \ln (1 - e^{-\beta \hbar \omega_{\mathbf{k}}}). \quad (9)$$

- Explain why the the average occupations numbers can be written as

$$\langle N(\mathbf{k}, r) \rangle = -\frac{1}{2} \frac{1}{\beta \hbar} \frac{\partial}{\partial \omega_{\mathbf{k}}} \ln Z. \quad (10)$$

- Find an explicit expression for $\langle N(\mathbf{k}, r) \rangle$.
- To evaluate many physical quantities explicitly in the limit $V \rightarrow \infty$ one makes the substitution

$$\sum_{\mathbf{k}, r} F(\mathbf{k}, r) \rightarrow V \mathcal{N} \sum_r \int d^3 \mathbf{k} F(\mathbf{k}, r), \quad (11)$$

valid for continuous functions $F(\mathbf{k}, r)$.

Explain the origin of this substitution. What is the dimensionless number $\mathcal{N}$?

- In most of the universe the photons have a temperature $T = 2.725$ K.
How many photons $N = \sum_{\mathbf{k}, r} \langle N(\mathbf{k}, r) \rangle$ are there on average per m^3 ?

Some physical constants, and an integral:

$$\hbar = 1.054\,571\,628 \times 10^{-34} \text{ Js}, \quad k_B = 1.380\,6503 \times 10^{-23} \text{ J/K}, \quad c = 299\,792\,458 \text{ m/s} \quad (12)$$

$$\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2\zeta(3) \approx 2.404 \dots \quad (13)$$