



Contact during the exam:
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Exam in TFY4230 STATISTICAL PHYSICS

Tuesday august 9th, 2011
09:00–13:00

Allowed help: Alternativ C

Standard calculator (according to list prepared by NTNU).

K. Rottman: *Matematisk formelsamling* (all languages).

Barnett & Cronin: *Mathematical Formulae*

Det er også en norsk versjon av dette eksamenssettet.

This problemset consists of 2 pages.

Problem 1. Chains of Ising spins

A cyclic chain of three Ising spins have the Hamiltonian

$$H = J (s_1 s_2 + s_2 s_3 + s_3 s_1). \quad (1)$$

- a) Write down all configurations and the corresponding energies for this chain.
- b) Find the partition function $Z = e^{-\beta F} = e^{-\beta U + S/k_B}$ for this chain.
- c) Find the mean energy $U = \langle H \rangle$ for this chain. Consider in particular the limit $T \rightarrow 0$, both for $J > 0$ and $J < 0$.
- d) Find the heat capacity $C = \frac{\partial U}{\partial T}$ of this chain.
- e) Find the entropy S of this chain. Consider in particular the limit $T \rightarrow 0$, both for $J > 0$ and $J < 0$.
- f) Find the correlation function $\langle s_1 s_2 \rangle$ for this chain.
- g) Sketch how you would do the corresponding analysis for a cyclic chain with N Ising spins, i.e. with Hamiltonian $H = J \left(s_N s_1 + \sum_{j=1}^{N-1} s_j s_{j+1} \right)$.

Problem 2. Quantum magnetism

The one-particle Hamiltonian for an electron (charge $q = -e$) in a magnetic field $\mathbf{B}$ is

$$H = \frac{1}{2m_e} (\mathbf{p} + e\mathbf{A})^2 - g\mu_B B s_z, \quad (2)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$. After quantization the eigenenergies of H are found to be

$$E = \frac{1}{2m_e} p_z^2 + \left(n + \frac{1}{2}\right) \varepsilon_a + s_z \varepsilon_b, \quad \text{where } s_z = \pm \frac{1}{2} \text{ and } n = 0, 1, \dots \quad (3)$$

Here $\varepsilon_a = \mu_B B$ and $\varepsilon_b = \frac{1}{2} g \mu_B B$. For each value of p_z , n , and s_z there are eBA/h degenerate states, where $\mathcal{A}$ is the area of the system normal to the magnetic field.

In the grand canonical ensemble the partition function for a gas of such electrons can be expressed as

$$\beta p = \frac{\ln \Xi}{V} = \frac{eB\sqrt{2m_e}}{h^2} \sum_{s_z = \pm \frac{1}{2}} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\varepsilon_z}{\sqrt{\varepsilon_z}} \ln \left\{ 1 + e^{-\beta[\varepsilon_z + (n+1/2)\varepsilon_a + s_z \varepsilon_b - \mu]} \right\}, \quad (4)$$

when we ignore the interaction between electrons.

- Sketch the general connection between one-particle states E_n and the grand canonical partition function Ξ for an ideal Fermi gas.
- Indicate how one in this particular case arrive at the partition function (4) from the one-particle states (3).
- Show that the partition function (4) can be rewritten in the form

$$\begin{aligned} \beta p = & \sum_{M=1}^{\infty} \frac{(-1)^{M+1}}{M} e^{M\beta\mu} \times \frac{eB\sqrt{2m_e}}{h^2} \times \\ & \times \sum_{s_z = \pm 1/2} e^{-s_z M\beta\varepsilon_b} \sum_{n=0}^{\infty} e^{-(n+1/2)M\beta\varepsilon_a} \int_0^{\infty} \frac{d\varepsilon_z}{\sqrt{\varepsilon_z}} e^{-M\beta\varepsilon_z}. \end{aligned} \quad (5)$$

- Perform the summations over s_z and n , and the integration over ε_z in equation (5).
- Consider the limit $B \rightarrow 0$ of your result from point **d**). Does the result look like the partition function for an ideal electron gas?
- The average magnetization per volume unit is here given by the expression

$$M = \left(\frac{\partial p}{\partial B} \right)_{\beta, \mu}. \quad (6)$$

Calculate this expression to first order in the fugacity $z = \lambda^{-3} e^{\beta\mu}$, where $\lambda = h^2 / \sqrt{2\pi k_B T m_e}$ is the thermal *de Broglie* wavelength of the electron. You may assume that the quantity $u \equiv \beta\mu_B B$ is small, and only calculate M to first order in u .

- For which values of g is the system *paramagnetic*, and for which values of g is it *diamagnetic*?

Given: Some of the expressions below *may* be of use for solving this exam set.

$$(1-x)^{-1} = \sum_{M=0}^{\infty} x^M, \quad (7)$$

$$\ln(1+x) = \sum_{M=1}^{\infty} \frac{(-1)^{M+1}}{M} x^M, \quad (8)$$

$$\int_0^{\infty} \frac{dt}{\sqrt{t}} e^{-t} = \sqrt{\pi}. \quad (9)$$