



Contact during the exam:
Professor Kåre Olaussen
Telephone: 9 36 52 eller 45 43 71 70

Exam in TFY4230 STATISTICAL PHYSICS

Friday december 19, 2014
09:00–13:00

Allowed help: Alternativ C

Standard calculator (according to the NTNU list).

One A4 formula sheet; personal handwritten notes are allowed on this.

K. Rottman: *Matematisk formelsamling* (all languages).

Barnett & Cronin: *Mathematical Formulae*.

This problemset consists of 3 pages.

Problem 1. Cold Fermi gases

The grand canonical partition function Θ for an ideal Fermi gas can be expressed in the form

$$\ln \Theta = Vg \int \frac{d^3k}{(2\pi)^3} \ln \left[1 + e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)} \right]. \quad (1)$$

- a) What are β and μ ? How is Θ identified with thermodynamical properties?
- b) What is g ? Does there exist physical systems where one may set $g = 1$?
- c) What is $\varepsilon_{\mathbf{k}}$? What is the form of $\varepsilon_{\mathbf{k}}$ for relativistic particles with mass?
- d) Find (or write down) an integral expression for the particle density $\rho = N/V$ of this system.
Sketch how the integrand of this expression varies with $\varepsilon_{\mathbf{k}}$. Assume that μ is positive and large; indicate in particular the limits $\beta(\varepsilon_{\mathbf{k}} - \mu) \ll -1$ and $\beta(\varepsilon_{\mathbf{k}} - \mu) \gg 1$.
- e) Find (or write down) an integral expression for the energy density E/V of this system.

Assume now that we may make the approximation $\varepsilon_{\mathbf{k}} - \mu \approx \frac{\hbar^2}{2m} \mathbf{k}^2 - \mu_{\text{NR}}$, and that the temperature may be set to zero.

- f) Find the connection between μ_{NR} (in this case also denoted the Fermi energy E_F) and the particle density ρ .
- g) Show that the answer of the previous point essentially can be found by dimensional analysis (apart for a numerical factor which must be of order 1):
 - (i) Which physical parameters can the answer depend on?
 - (ii) How must these parameters be combined to give an expression of dimension energy?

- h) The number density of free electrons in aluminium is $\rho = 2.1 \cdot 10^{29} \text{ m}^{-3}$. Assume that these electrons can be treated as an ideal Fermi gas.

What is the Fermi temperature T_F (defined by $k_B T_F = E_F$) for this material?

Given:

$$\begin{aligned}\hbar &= 1.054 \cdot 10^{-34} \text{ kg m}^2 \text{ s}^{-1}, \\ k_B &= 1.381 \cdot 10^{-23} \text{ kg m}^2 \text{ s}^{-2} \text{ K}^{-1}, \\ m_e &= 9.109 \cdot 10^{-31} \text{ kg}, \\ m_{\text{Al}} &\approx 4.482 \cdot 10^{-26} \text{ kg}.\end{aligned}$$

Problem 2. Frustrated Ising chain

In this problem we consider a closed Ising chain with antiferromagnetic interactions, with an odd number of spins. We first simplify to the case of three spins. The Hamilton function then becomes

$$H = J(s_0 s_1 + s_1 s_2 + s_2 s_0), \quad (2)$$

where each Ising spin s_i takes the values ± 1 , and $J > 0$.

- What are the possible energies of this chain, and how many configurations are there for each of these energies?
- What is the entropy S of this chain at (i) zero temperature, and (ii) infinite temperature?
- Write down the canonical partition function for this chain.
- Calculate the mean energy E as function of the temperature parameter β .
- Calculate the heat capacity C as function of the temperature parameter β .
- What is the behavior of C as (i) $T \rightarrow 0$, and (ii) as $T \rightarrow \infty$?
- Calculate the entropy S of this chain as function of the temperature parameter β .

Now add a magnetic field, such that the Hamilton function acquires an additional contribution,

$$\Delta H = -\mu B(s_0 + s_1 + s_2). \quad (3)$$

- What are the possible energies of this chain now; and how many configurations are there for each of these energies?

Finally consider the general case of a chain with $2N + 1$ spins.

- What is the *lowest* possible energy of the chain, and how many configurations have this energy?

Problem 3. Lattice vibrations

A slightly simplified model for linear lattice vibrations is given by the Hamilton function

$$H = \frac{1}{2M} \sum_m p_m^2 + \frac{1}{2} \sum_{mn} q_m K_{m,n} q_n, \quad (4)$$

where $K_{m,n}$ is a symmetric $N \times N$ matrix with all eigenvalues positive, $\lambda_j = M \omega_j^2$.

- What is the classical heat capacity C for this system according to the equipartition principle?

Quantum mechanically the system constitutes a collection of N harmonic oscillators, with frequencies ω_j defined by the eigenvalues

$$\mathcal{S} = \{\lambda_j = M\omega_j^2 \mid j = 0, \dots, N-1\}. \quad (5)$$

To find $\mathcal{S}$ numerically for a large system one may use a routine from **scipy.sparse.linalg**. This requires one to make a function which performs the operation $\psi_m \rightarrow \sum_n K_{mn}\psi_n$. The code below shows a one-dimensional example of such an operation.

```
1 def K(psi):
2     """Return (-1) times the 1D lattice laplacian of psi"""
3     Kpsi = 2*psi
4     Kpsi -= numpy.roll(psi, 1, axis=0)
5     Kpsi -= numpy.roll(psi, -1, axis=0)
6     return Kpsi/2
```

- b) From the code above one may read out what the explicit expression for $\sum_{mn} q_m K_{m,n} q_n$ is in this case. Write down this expression.
- c) Generalize the code above to two- and three-dimensional lattices with corresponding nearest-neighbor interactions.

For a large lattice the frequencies will be very close, and are best described by a function $\rho(\omega)$, the density of states.

Numerically we may construct such a density through a histogram. The code below is an example of how this can be done (taken from a slightly different situation).

```
1 def makeHistogram(data, min, max, nbins):
2     """Return a histogram of the contents in data"""
3     bins = numpy.linspace(min, max, nbins+1)
4     return numpy.histogram(data, bins=bins)
```

- d) Modify the code above to the calculation of $\rho(\omega)$, under the assumption that **data** contains the set $\mathcal{S}$ of eigenvalues.
- e) Assume that $\mathcal{S}$ consists of 10 000 eigenvalues in the interval between 0 and 10. Which values would you choose for the parameters **min**, **max** and **nbins**?